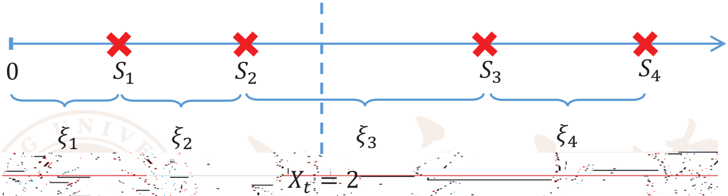
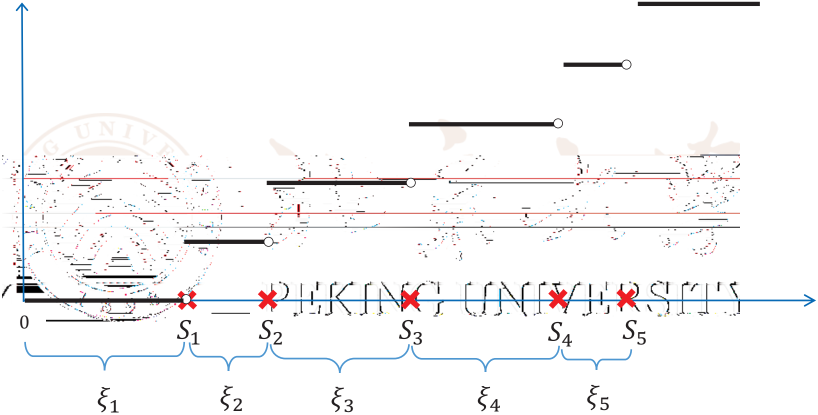
第二章、跳过程

§2.1 泊松过程(Poisson Process)



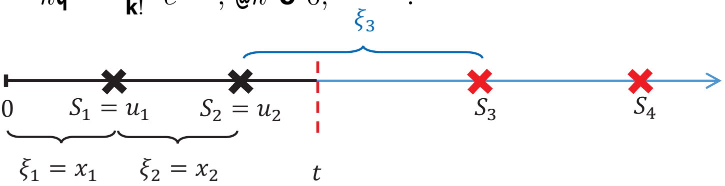
- $\xi_1, \xi_2, \ldots$ i.i.d. $\sim \mathrm{Exp}(\lambda)$. $P(\xi > t) = e^{-\lambda t}$.
- $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$, 其中 $S_0 := 0$.
-

轨道图: $X_t := |\Xi \cap [0,t]|$, $\Xi = \{S_1, S_2, \ldots\}$.



- $\forall t \geq 0$, $P(t \in \Xi) = P(\bigcup_n \{S_n = t\}) \leq \sum_{n=1}^{\infty} P(S_n = t) = 0$.

验证$P(Z=k)=\frac{(\lambda t)^k}{k!}e^{-\lambda t}$, $\forall k\ge 0$; "……".



- 假设$k\ge 1$, $0<u_1<\dots<u_k<t$,

  $P(S_1\in u_1,\dots,S_k\in u_k)=p_{S_1,\dots,S_k}(u_1,\dots,u_k)du_1\cdots du_k$.

- $P(Z=k;A)=P(A;\xi_{k+1}>t-S_k)=e^{-\lambda t}\cdot e^{\lambda u_k}P(A)$,

  $p_{S_1,\dots,S_k}(u_1,\dots,u_k)=\lambda^k e^{-\lambda(x_1+\dots+x_k)}=\lambda^k e^{-\lambda u_k}$

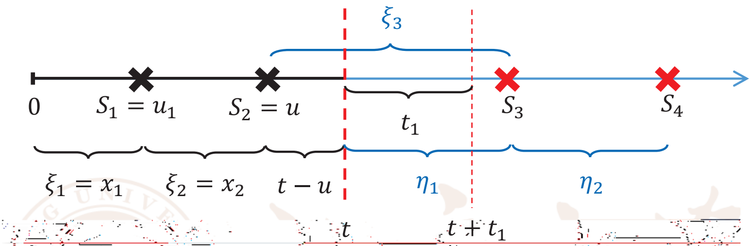
- $P(Z=k;A)=\lambda^k e^{-\lambda t}1_{\{0<u_1<\dots<u_k<t\}}du_1\cdots du_k$, $(\Rightarrow \frac{t^k}{k!})$.

- $Z\sim\mathcal{P}(\lambda t)$, $P(A\mid Z=k)=\frac{k!}{t^k}1_{\{0<u_1<\dots<u_k<t\}}du_1\cdots du_k$.

- $Z\overset{d}{=}\{U_1,\dots,U_W\}$, 其中$U_1,U_2,\dots$ i.i.d., $\sim U(0,t)$,

  $W\sim\mathcal{P}(\lambda t)$, 且所有随机变量相互独立.

验证 $Y \sim PP(\lambda)$, 且 Y, Z 相互独立.



- 令 $B = \{\eta_1 > t_1, \eta_2 > t_2, \ldots, \eta_m > t_m\}$. 往验证

$$P(|Z| = k, A; B) = P(|Z| = k, A) \cdot e^{-\lambda(t_1+\ldots+t_m)}.$$

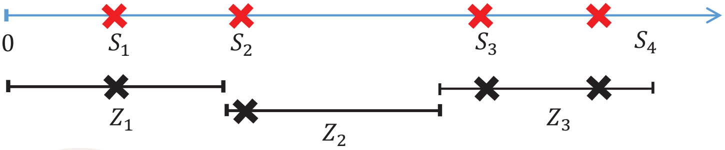
- 令 $\hat{B} = \{S_{k+1} > t + t_1; \xi_{k+2} > t_2, \ldots, \xi_{k+m} > t_m\}$. 则

$$\{|Z| = k; B\} = \{|Z| = k; \hat{B}\}.$$

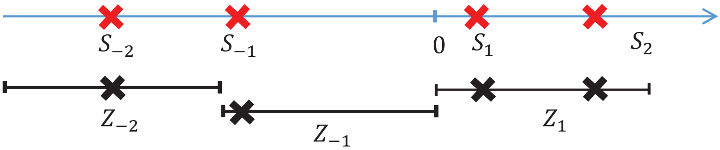
- 左边 $= P(A, S_{k+1} > t; S_{k+1} > t + t_1, \tilde{B})$

$$= P(|X_{t+t_1}| = k, A; \tilde{B}) = P(A; |Z| = k) \cdot e^{-\lambda t_1} \cdot P(\tilde{B}).$$

构造泊松流(命题2.1.10):

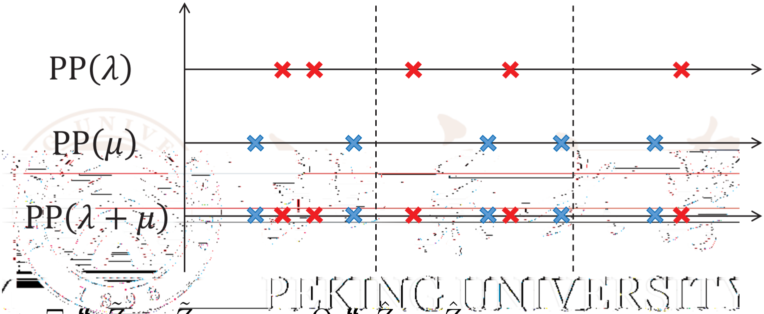


- Z_1 “ Ξ **X** r0, 1s, θ “ θ_1 : Ξ **Ñ** Y.
- Z_2 “ θpΞq **X** r0, 1s, Z_3 “ $\theta^{(2)}$pΞq **X** r0, 1s, . . .
- $Z_1, Z_2, \ldots$ i.i.d., Ξ “ Z_1 _ Z_2 _ . . . .
- $Z_{\mathbf{n}}$, n **P** $\mathbb{Z}$, i.i.d., Ξ “ . . . _ Z_{-2} _ Z_{-1} _ Z_1 _ Z_2 _ . . . .



泊松流的合并与细分:

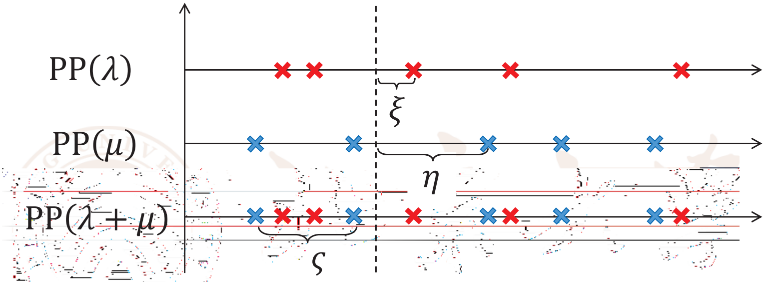
$\Xi \sim \mathrm{PP}(\lambda)$ 与 $\Theta \sim \mathrm{PP}(\mu)$ 相互独立 vs $\Xi \cup \Theta \sim \mathrm{PF}(\lambda + \mu)$.



- $\Xi = \{\tilde{Z}_1 < \tilde{Z}_2 < \ldots\}$, $\Theta = \{\hat{Z}_1 < \hat{Z}_2 < \ldots\}$, $Z_n \overset{d}{=} \{\tilde{U}_1, \ldots, \tilde{U}_{\tilde{W}}\}$, $\hat{Z}_n \overset{d}{=} \{\hat{U}_1, \ldots, \hat{U}_{\hat{W}}\}$,
- $Z_n = \tilde{Z}_n \cup \hat{Z}_n$, $\tilde{Z}_1 \overset{d}{=} \{U_1, \ldots, U_W\}$, $W = \tilde{W} + \hat{W}$.
- $\Xi \cup \Theta = \{Z_1 < Z_2 < \ldots\} \sim \mathrm{PP}(\lambda + \mu)$.

泊松流的合并与细分:

Ξ „ PPpλq 与 Θ „ PPpμq 相互独立 vs Ξ Y Θ „ PFpλ ` μq.



- ζ “ ξ ^ η:

$$P\text{p}\zeta \text{ ą } t\text{q} \text{ “ } e^{-\lambda t} \text{ ^ } e^{-\mu t}, \quad P\text{p}\xi \text{ ă } \eta\text{q} \text{ “ } \frac{\lambda}{\lambda \text{ ` } \mu} \text{ “ } p.$$

- ζ „ Exppλ ` μq, V „ $\mathcal{G}$pp q.

$$\zeta_1 \text{ ` } \cdots \text{ ` } \zeta_V \text{ „ Expp}\lambda\text{q}.$$

泊松点过程Ξ:

- $\mathbb{R}^d$ 上: (i) $|\Xi \cap D| \sim \mathcal{P}(\lambda|D|)$,
  (ii) 若$D_1,\dots,D_n$ 互不交, 则$|\Xi\cap D_i|$, $i=1,\dots,n$ 独立.
- $(\mathbb{S},\mu(\cdot))$ 上: (i) $|\Xi\cap D|\sim\mathcal{P}(\mu(D))$, (ii).
- 构造:
  (a) 将$\mathbb{S}$划分为$D_1, D_2,\cdots$ 使得$0<\mu(D_n)<\infty$;
  (b) 取$U_{n1},U_{n2},\dots \overset{\text{i.i.d.}}{\sim} \frac{1}{\mu(D_n)}\mu(\cdot)$, $W_n\sim\mathcal{P}(\mu(D_n))$;
  (c) $\Xi = \bigcup_n \{U_{n1},\dots,U_{nW_n}\}$.
- 例: $\mathbb{R}_1$ 上, $\mu((a,b)) = \int_a^b f(x)dx$. 则

$$P(\Xi\cap[a,b]=\emptyset) = e^{-\int_a^b f(x)dx}.$$

§2.2 跳过程的构造及其转移概率

1. 定义

- $\forall i \in S$,

  一组小闹钟q_{ij}, $j \neq i$; 或

  大闹钟q_i 与色子$\hat{p}_{ij}$, $j \neq i$:

$$q_i = \sum_{j \neq i} q_{ij} < \infty, \quad \hat{p}_{ij} = \frac{q_{ij}}{q_i}.$$

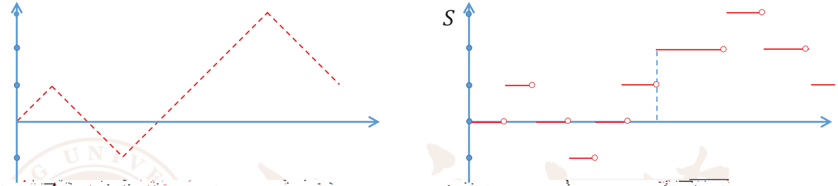
- 特殊情况: 吸收态: $q_i = 0$, $\hat{p}_{ii} := 1$.

  非吸收态: $q_i > 0$, $\hat{p}_{ii} := 0$.

- 速率矩阵(定义2.2.1):

$$Q = (q_{ij})_{S \times S}, \quad q_{ii} := -q_i.$$

- 色子 $\hat{P} = (\hat{p}_{ij})_{S\times S}$: 嵌入链 $\{\hat{X}_n\}$;　　闹钟: 时间变换.

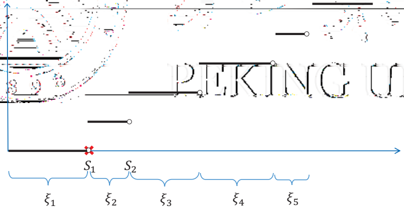


2. 爆炸与非爆炸.

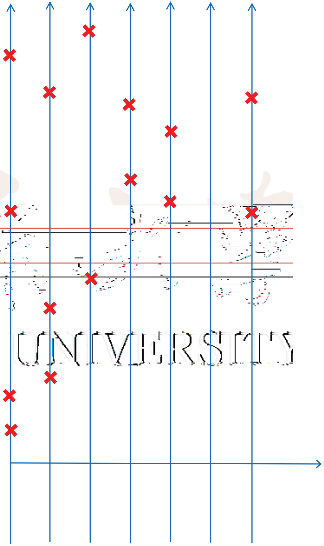
- 命题2.2.3. $\zeta_1, \zeta_2, \cdots$ 独立, $\zeta_n \sim \mathrm{Exp}(\lambda_n)$, $\tau = \sum_n \zeta_n$. 则

$$\sum_i \frac{1}{\lambda_i} = \infty \quad \text{iff} \quad P(\tau = \infty) = 1,$$

$$\sum_i \frac{1}{\lambda_i} < \infty \quad \text{iff} \quad P(\tau < \infty) = 1.$$



图表示与对偶:



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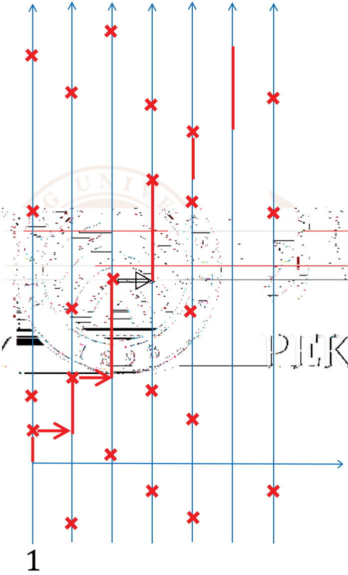
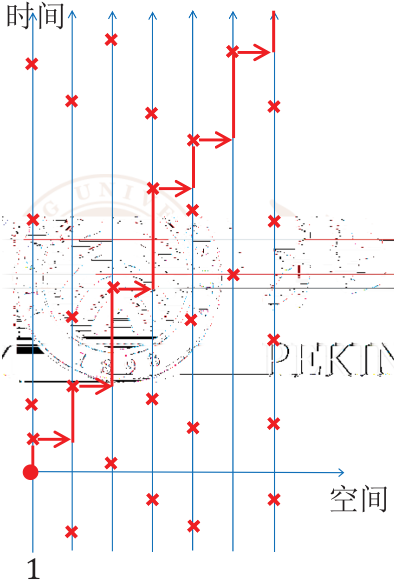
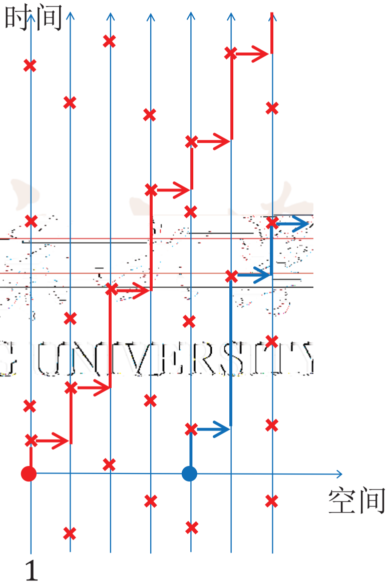
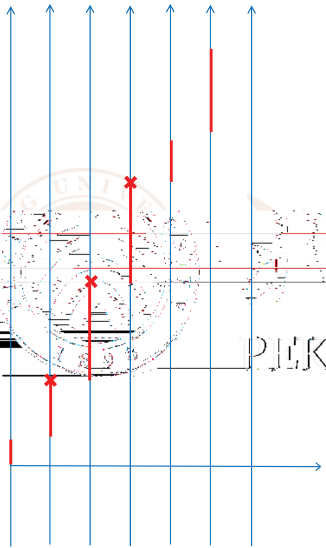


图表示与对偶:



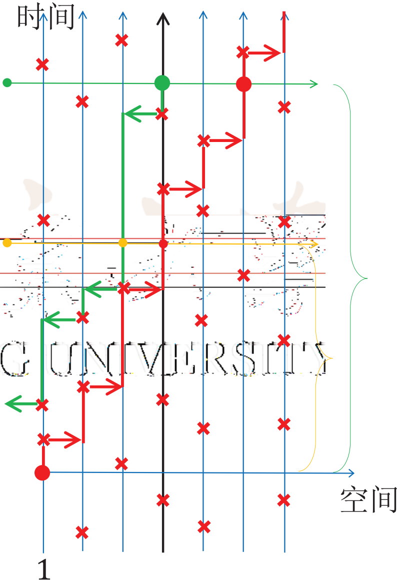


图表示与对偶:



图表示与对偶:

- $X_t \geq i + 1$ iff $Y_t^{(i)} = 0$.
- 概率 $= (1 - e^{-\lambda t})^i$.



3. 转移概率与转移速率.

- 转移概率 $\mathbf{P}(t) = (p_{ij}(t))_{S\times S}$:

$$p_{ij}(t) = P_i(X_t = j) = \sum_{n=0}^{\infty} P_i(X_t = j, N_t = n).$$

- $P_i(\hat{X}_1 = i_1, \ldots, \hat{X}_n = i_n; S_n \le t < S_{n+1})$
- $= p_{i_0 i_1}\cdots p_{i_{n-1} i_n} P(\cdots)$,
- $= \int_{\Delta_n} q_{i_0} e^{-q_{i_0}x_0} \cdots q_{i_n} e^{-q_{i_n}x_n} dx_0 \cdots dx_n$,

$\Delta_n := \{x : x_i \ge 0, \forall i; x_0 + \cdots + x_{n-1} \le t < x_0 + \cdots + x_n\}$.

- 非爆炸: $\sum_j p_{ij}(t) = 1, \forall i, t$.

$$\sum_{j\in S} p_{ij}(t) = P_i(X_t \in S) = P_i(\tau_\infty > t) \le 1.$$

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- 后退方程$\mathbf{P}'ptq$ “ $\mathbf{Q}\mathbf{P}ptq$ 的应用. f 是.

补充知识:

- 定义2.2.19. (强连续)马氏半群$\{P(t): t \ge 0\}$:

  (1) $P(0) = I$,　(2) $P(t+s) = P(t)P(s)$,

  (3) $\lim_{t\to 0} P(t) = P(0)$, $p_{ij}(t) \to p_{ij}(0)$, $\forall i, j$.

- 命题2.2.22. $Q = P'(0)$ 存在且满足:

  $0 \le q_{ij} < \infty$;　$0 \le q_i := -q_{ii} \le \infty$;　$\sum_{j\ne i} q_{ij} \le q_i$.

- 命题2.2.24. Q 保守，则后退方程$P'(t) = QP(t)$ 成立.

- 连续时间马氏链$\{X_t\}$: 以$\{P(t), t \ge 0\}$为转移矩阵.

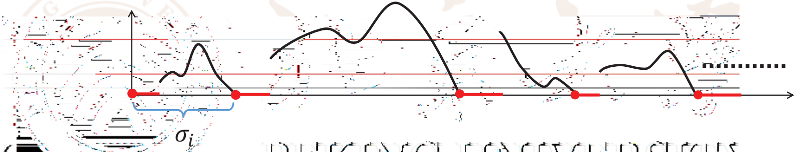
- 生成元: 定义域为$\mathcal{D} = \mathcal{D}(\mathcal{L}) \subset 2^S$,

$$\mathcal{L}: \mathcal{D} \to 2^S, \quad f \mapsto \mathcal{L}(f) = Qf,$$

$$\frac{d}{dt} f_t = \mathcal{L}(f_t), \quad f_t(i) := E_i f(X_t).$$

§2.3 首达时、吸收概率

- 不可约: 可达、互通. (定义2.3.1, 命题2.3.2)
- 强马氏性(引理2.3.3 & 2.4.2). 令$\tau = \tau_i$ 或σ_i. 在$\tau < \infty$ 的条件下, $\{Y_t := X_{\tau+t}\}$ 为从i 出发的跳过程, 且与$X_{[0,\tau]}$ 独立.



- $x_i = P_i(\tau_0 < \infty)$: $x_i = \sum_j p_{ij} x_j$, $i \neq 0$; $x_0 = 1$.
- $y_i = E_i \tau_0$: $y_i = \frac{1}{q_i} + \sum_j p_{ij} y_j$, $i \neq 0$; $y_0 = 0$.
- $z_i = E_i \int_0^{\tau_D} 1_{\{X_t = o\}} dt$:

  $z_i = \sum_j p_{ij} z_j$, $i \in D$, $i \neq k$; $z_o = \frac{1}{q_o} + \sum_i p_{oi} z_i$; $z|_D = 0$.

- 常返: $P_i(\forall t, \exists s > t, \text{ s.t. } X_s = i) = 1$.
- 命题2.4.5. i 常返的等价条件:

  (a) $q_i = 0$ 或 $P_i(\sigma_i < \infty) = 1$,

  (b) 嵌入链 $\{\hat{X}_n\}$ 常返,

  (c) 格林函数发散, 即 $G_{ii} = \int p_{ii}(t)dt = \infty$,

  (d) 骨架链 $\{X_{n\delta} : n = 0, 1, 2 \cdots\}$ 常返.

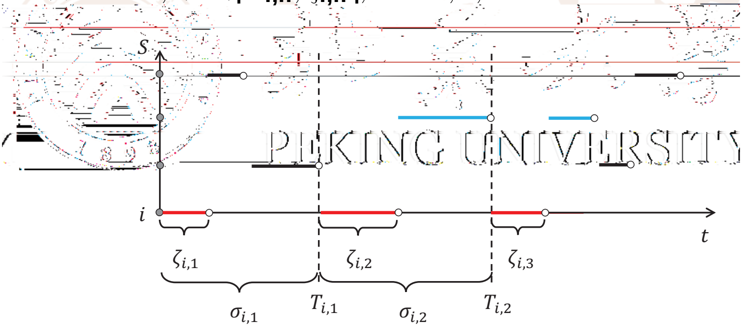
- 推论2.4.3 & 2.4.4. 在i的总耗时:

$$\zeta_1 + \cdots + \zeta_{\hat{V}_i}, \qquad \zeta_n \text{ 独立地 } \sim \mathrm{Exp}(q_i).$$

因此, 常返 $\Rightarrow$ 非爆炸, $G_{ij} = \frac{1}{q_j}\hat{G}_{ij}$.

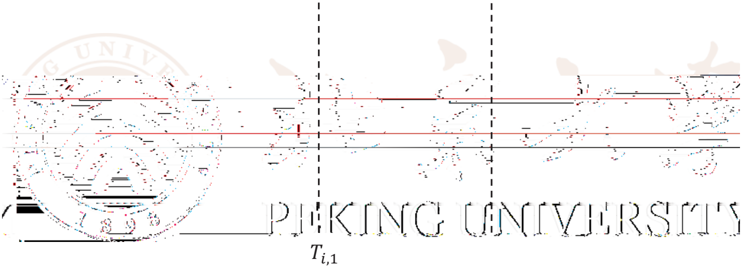
§2.5 不变分布与正常返

- 定义2.5.1. 不变分布/不变测度: $\pi = \pi \mathbf{P}(t)$, $\forall t$.
- 引理2.5.3. 若不变分布存在, 则非爆炸.
- 强马氏性(引理2.5.5): 假设i 常返, $q_i > 0$. 假设$X_0 = i$. 那么, 游弋i.i.d., 从而$(\sigma_{i,n}, \zeta_{i,n})$, $n = 1, 2, \cdots$ i.i.d.



- 命题2.5.8. 假设i 常返且$q_i > 0$. 令$V_i(t) = \int_0^t 1_{\{X_s=i\}} ds$. 则

$$P\left(\lim_{t\to\infty} \frac{1}{t} V_i(t) = \frac{1}{q_i E_i \sigma_i}\right) = 1.$$

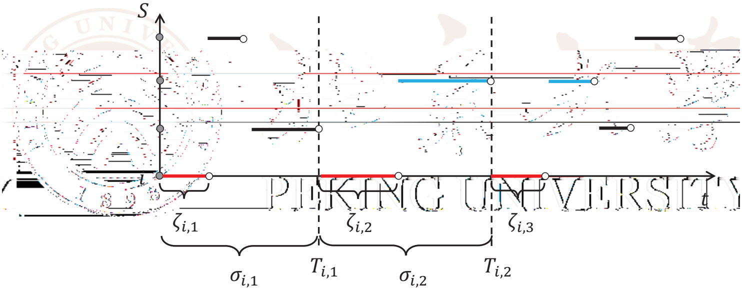


命题 (命题2.5.8)

设不可约、常返. 则$q_i\mu_i = \hat{\mu}_i$; $\mu Q = 0$; $\lambda Q = 0$ iff $\lambda = c\mu$.

- $\mu_i = E_0\int_0^{\sigma} 1_{\{X_t=i\}}dt = \frac{1}{q_i} \wedge E_0 V_i^{(Y)}$, $\forall i$. $\sigma = \sigma_0$.

  $\Leftarrow = \zeta_1 + \cdots + \zeta_{\tilde{V}_i}$, $\zeta_n \sim \mathrm{Exp}(q_i)$.



- $\frac{1}{q_i}q_{ij} = \hat{p}_{ij}$, $\lambda Q = 0$ iff

$$(\lambda_j q_j) = \sum_{i\neq j}\lambda_i q_{ij} = \sum_{i\neq j}(\lambda_i q_i)\hat{p}_{ij},\ \text{i.e., } q\lambda = c\hat{\mu}.$$

命题 (命题2.5.9, 推论2.5.10)

- 定义2.5.7. 正常返: $q_i = 0$ 或 $E_i\sigma_i < \infty$.

  常返: $q_i > 0$, $P_i(\sigma_i < \infty) = 1$, 且 $E_i\sigma_i = \infty$.

- 命题2.5.11. 设不可约. 则下面三条等价: (i) 所有状态正常返, (2) 存在正常返态, (3) 存在不变分布. 此时, $\pi Q = 0$,

$$\pi_i = \frac{1}{q_i E_i\sigma_i} = \frac{1}{E_o\sigma_o} E_o\int_0^{\sigma_o} 1_{\{X_t=i\}}\,dt \approx i \text{ 的频率}.$$

- 例2.5.4. $\pi Q = 0$, 但 π 不是不变分布.

- 遍历(定理2.5.17): 不可约、正常返, $\sum_i \pi_i|f(i)| < \infty$, 则

$$P_\mu\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X_s)\,ds = \sum_{i\in S}\pi_i f(i)\right) = 1.$$

- 强遍历(定理2.5.18): $\lim_{t\to\infty} p_{ij}(t) = \pi_j$, $\forall j$.

§2.6 可逆分布

- 总假设Q 不可约. 若π 为测度, 满足$\pi\mathbf{Q} = 0$, 令

$$\tilde q_{ij} := \frac{\pi_j q_{ji}}{\pi_i}.$$

则, $\pi_i\tilde q_{ij} = \pi_j q_{ji}$, $\forall i,j$.

- $\tilde{\mathbf{Q}} = (\tilde q_{ij})_{S\times S}$ 仍为转移矩阵, 且$\tilde q_i = q_i$, $\pi\tilde{\mathbf{Q}} = 0$.
-