

1. 广义似然比检验的思想

§8.4 广义似然比检验和关于正态总体参数的检验

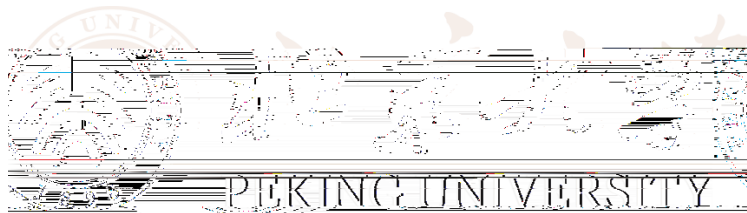
- 假设检验问题 $H_0: \theta \in \Theta_0 \leftrightarrow H_1: \theta \in \Theta_1$.
- 考虑 θ 分别在 $\Theta = \Theta_0 \cup \Theta_1$ 与 Θ_0 中的最大似然估计 $\hat{\theta} \in \Theta$ 与 $\hat{\theta}_0 \in \Theta_0$:

$$L(\vec{x}, \hat{\theta}) = \sup_{\theta \in \Theta} L(\vec{x}, \theta), \quad L(\vec{x}, \hat{\theta}_0) = \sup_{\theta \in \Theta_0} L(\vec{x}, \theta).$$

- 定义4.1 称 $\lambda(\vec{x}) = L(\vec{x}, \hat{\theta}) / L(\vec{x}, \hat{\theta}_0)$ 为广义似然比.
- 广义似然比否定域指

$$\mathcal{W} := \left\{ \vec{x} : \frac{L(\vec{x}, \hat{\theta})}{L(\vec{x}, \hat{\theta}_0)} > c \right\} = \{ \vec{x} : \lambda(\vec{x}) > c \},$$

其中 $c \geq 1$, 且满足 $\sup_{\theta \in \Theta_0} P_{\theta}(\vec{X} \in \mathcal{W}) = \alpha$.



A. 单边问题 $H_0: \mu \leq \mu_0 \leftrightarrow H_1: \mu > \mu_0$ (续).

- 广义似然比: $\lambda(\vec{x}) = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}\right)^{\frac{n}{2}}$, 其中,

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2, \quad \hat{\sigma}_0^2 = \begin{cases} \hat{\sigma}^2, & \text{若 } \bar{x} \leq \mu_0, \\ \frac{1}{n} \sum_{i=1}^n (x_i - \mu_0)^2, & \text{若 } \bar{x} > \mu_0, \end{cases}$$

- 广义似然比否定域: $c_1 \geq 1$,

$$W = \left\{ \bar{x} : \frac{\hat{\sigma}_0^2}{\hat{\sigma}^2} > c_1 \right\} = \left\{ \bar{x} : \bar{x} > \mu_0 \text{ 且 } \frac{\sum_{i=1}^n (x_i - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2} > c_1 \right\}.$$

- $\sum_{i=1}^n (x_i - \mu_0)^2 = \sum_{i=1}^n (x_i - \bar{x})^2 + n(\mu_0 - \bar{x})^2$, 因此

$$\frac{\sum_{i=1}^n (x_i - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2} = 1 + \frac{T^2}{n-1}, \quad \text{其中 } T = \frac{\sqrt{n}(\bar{x} - \mu_0)}{S}.$$

A. 单边问题 $H_0: \mu \leq \mu_0 \leftrightarrow H_1: \mu > \mu_0$ (续).

• 广义似然比: $\mathcal{W} = \left\{ \vec{x} : \bar{x} > \mu_0 \text{ 且 } \frac{\sum_{i=1}^n (x_i - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2} > c_1 \right\}$. 其中,

$$\frac{\sum_{i=1}^n (x_i - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2} = 1 + \frac{T^2}{n-1}, \quad \text{其中 } T = \frac{\sqrt{n}(\bar{x} - \mu_0)}{S}.$$

• 总结: $c > 0$,

$$\mathcal{W} = \{ \vec{x} : T > 0 \text{ 且 } T^2 > c_2 \} = \{ \vec{x} : T > c \}.$$

• 根据 α 求 c :

$\forall \mu \leq \mu_0, T \leq \frac{\sqrt{n}(\bar{x} - \mu)}{S} =: T_{n-1} \sim t(n-1)$, 在 $\mu = \mu_0$ 时等号成立. 因此, 取 $c = t_{1-\alpha}(n-1)$ 即可满足

$$\max_{\mu \leq \mu_0} P_{\mu}(T > c) = P(T_{n-1} > c) = \alpha.$$

B. 双边问题 $H_0: \mu = \mu_0 \leftrightarrow H_1: \mu \neq \mu_0$.

- $\theta = (\mu, \sigma^2)$, $\Theta = (-\infty, \infty) \times (0, \infty)$, $\Theta_0 = \{\mu_0\} \times (0, \infty)$.
- 最大似然估计:

$$\hat{\mu} = \bar{x}, \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \hat{\mu})^2, \quad L(\vec{x}, \hat{\theta}) = (2\pi\hat{\sigma}^2)^{-\frac{n}{2}} e^{-\frac{n}{2}}$$

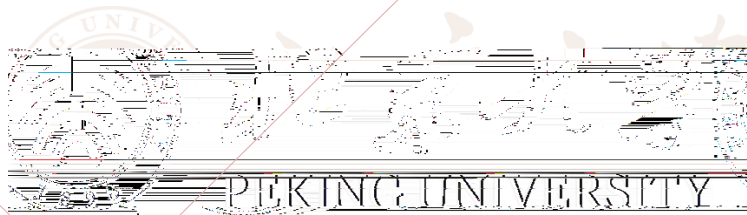
$$\hat{\mu} = \mu_0, \quad \hat{\sigma}_0^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \mu_0)^2, \quad L(\vec{x}, \hat{\theta}_0) = (2\pi\hat{\sigma}_0^2)^{-\frac{n}{2}} e^{-\frac{n}{2}}.$$

- 广义似然比否定域: 记 $T = \frac{\sqrt{n}(\bar{x} - \mu_0)}{\hat{\sigma}_0}$, 则

$$W = \left\{ \vec{x} : \frac{\hat{\sigma}_0^2}{\hat{\sigma}^2} > \tilde{c} \right\} = \{ \vec{x} : |T| > c \}.$$

- 根据 α 求 c : 取 $c = t_{1-\alpha/2}(n-1)$ 即可满足

$$P_{\mu_0}(\vec{X} \in W) = P_{\mu_0}(|T| > c) = \alpha.$$



A. 双边问题 $H_0: \sigma^2 = \sigma_0^2 \leftrightarrow H_1: \sigma^2 \neq \sigma_0^2$ (续).

• 广义似然比:

$$\lambda(\vec{x}) = u^{-\frac{n}{2}} e^{\frac{u}{2}} \left(\frac{e}{n}\right)^{-\frac{n}{2}}, \quad \text{其中 } u = u(\vec{x}) = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{\sigma_0^2}.$$

• $f(u) = \left(\frac{u}{n}\right)^{-\frac{n}{2}} e^{\frac{u}{2}}$ 关于 u 先↓后↑, (导函数先负后正);

• 若 $\sigma^2 = \sigma_0^2$, 则 $U = U(\vec{X}) = \frac{n\hat{\sigma}^2}{\sigma_0^2} \sim \chi^2(n-1)$.

• 广义似然比否定域

$$\{\vec{x} : f(u(\vec{x})) > c\} = \{\vec{x} : u(\vec{x}) < c_1 \text{ 或 } u(\vec{x}) > c_2\},$$

其中, c_1, c_2 满足 $f(c_1) = f(c_2) = c$ 且 $c_1 < c_2$.

A. 双边问题 $H_0: \sigma^2 = \sigma_0^2 \leftrightarrow H_1: \sigma^2 \neq \sigma_0^2$ (续).

- UMPU 否定域: 令 $g(u) := \left(\frac{u}{n}\right)^{-\frac{n-1}{2}} e^{\frac{u}{2}}$,

$$\{\vec{x} : g(u(\vec{x})) > c\} = \{\vec{x} : u(\vec{x}) < c_3 \text{ 或 } u(\vec{x}) > c_4\},$$

其中, c_3, c_4 满足 $g(c_3) = g(c_4) = c$ 且 $c_3 < c_4$,

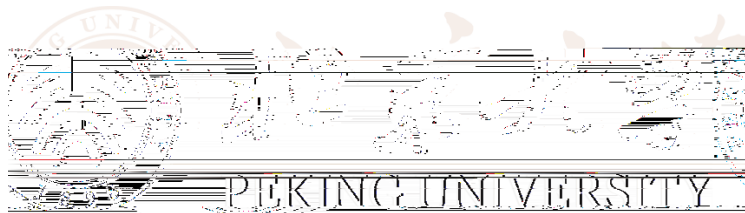
$$u(\vec{x}) = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{\sigma_0^2}.$$

- 根据 α 求 c 在 H_0 中, $U \in (c_3, c_4)$ 的概率为 α .

找 c 使得 $P_{\sigma_0^2}(U < c_3) + P_{\sigma_0^2}(U > c_4) = \alpha$.

- 实际操作: 取 $c_5 = \chi_{\alpha/2}^2(n-1)$, $c_6 = \chi^2$

B.



B. 单边问题 $H_0: \sigma^2 \geq \sigma_0^2 \leftrightarrow H_1: \sigma^2 < \sigma_0^2$ (续).

• 广义似然比否定域:

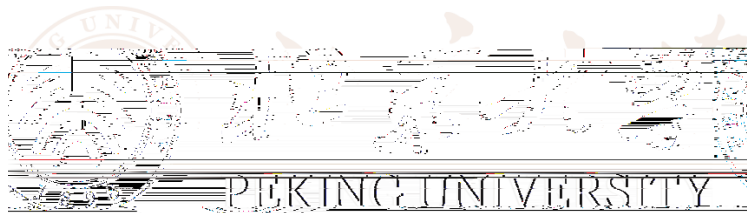
$$W = \{ \vec{x} : \hat{\sigma}^2 \leq \sigma_0^2, \left(\frac{u}{n} \right)^{-\frac{n}{2}} e^{\frac{u}{2}} > \tilde{c} \} = \{ \vec{x} : u < c \}.$$

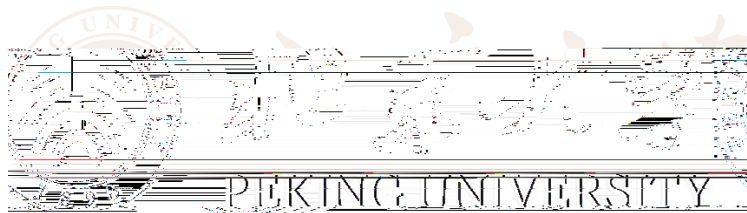
其中 $u \equiv u(\vec{x}) \equiv \frac{n\hat{\sigma}^2}{\sigma_0^2}$, $c < n$.

• 根据 α 求 c .

$$\forall \sigma^2 \geq \sigma_0^2, U := u(\vec{X}) \geq \frac{n\sigma^2}{\sigma_0^2} \equiv: U_{n-1} \sim \chi^2(n-1)$$

在 $\sigma^2 = \sigma_0^2$ 时, 等号成立.





A. 方差检验. $H_0: \sigma_1^2 = \sigma_2^2 \leftrightarrow H_1: \sigma_1^2 \neq \sigma_2^2$ (续).

• 在 Θ_0 中的最大似然估计:

• 似然函数 $L(\vec{x}, \vec{y}, \mu_1, \mu_2, \sigma^2)$:

$$\left(\frac{1}{\sqrt{2\pi\sigma}} \right)^{n_1+n_2} \exp \left\{ -\frac{1}{2\sigma^2} \left(\sum_{i=1}^{n_1} (x_i - \mu_1)^2 + \sum_{i=1}^{n_2} (y_i - \mu_2)^2 \right) \right\}.$$

• 似然估计:

$$\hat{\mu}_1 = \bar{x}, \quad \hat{\mu}_2 = \bar{y}, \quad \hat{\sigma}^2 = \frac{\sum_{i=1}^{n_1} (x_i - \bar{x})^2 + \sum_{i=1}^{n_2} (y_i - \bar{y})^2}{n_1 + n_2}.$$

• 将似然估计代入似然函数:

$$\hat{L}_0 = \left(\frac{1}{\sqrt{2\pi}} \right)^{n_1+n_2} \left(\frac{1}{\frac{1}{n_1+n_2}(u+v)} \right)^{\frac{n_1+n_2}{2}} e^{-\frac{n_1+n_2}{2}}.$$

A. 方差检验. $H_0 : \sigma_1^2 = \sigma_2^2 \leftrightarrow H_1 : \sigma_1^2 \neq \sigma_2^2$ (续).

• 广义似然比: $u = \sum_{i=1}^{n_1} (x_i - \bar{x})^2$, $v = \sum_{i=1}^{n_2} (y_i - \bar{y})^2$,

$$\lambda(\vec{x}, \vec{y}) = \frac{\left(\frac{1}{n_1+n_2} (u+v) \right)^{\frac{n_1+n_2}{2}}}{\left(\frac{1}{n_1} u \right)^{\frac{n_1}{2}} \left(\frac{1}{n_2} v \right)^{\frac{n_2}{2}}} = c_0 \left(\frac{u}{u+v} \right)^{-\frac{n_1}{2}} \left(\frac{v}{u+v} \right)^{-\frac{n_2}{2}}.$$

• 广义似然比否定域形如

$$\left\{ (\vec{x}, \vec{y}) : \frac{u}{u+v} < c_1 \text{ 或 } > c_2 \right\}.$$

其中, c_1, c_2 基本上无法计算.

• 实际操作: 取 c_3, c_4 使得

$$P_{H_0} \left(\frac{U}{U+V} < c_3 \right) = P_{H_0} \left(\frac{U}{U+V} > c_4 \right) = \frac{\alpha}{2}.$$

A. 方差检验. $H_0: \sigma_1^2 = \sigma_2^2 \leftrightarrow H_1: \sigma_1^2 \neq \sigma_2^2$ (续).

- 定义4.2 (F 分布). $F(m_1, m_2)$ 指 $\frac{K_{m_1}/m_1}{K_{m_2}/m_2}$ 的分布, 其中, $K_{m_1} \sim \chi^2(m_1)$, $K_{m_2} \sim \chi^2(m_2)$, 且 K_{m_1}, K_{m_2} 相互独立. (密度可根据定理3.4.2计算得到.)

- $U = \sum_{i=1}^{n_1} (X_i - \bar{X})^2, \quad U/\sigma_1^2 \sim \chi^2(n_1 - 1),$

- $V = \sum_{j=1}^{n_2} (Y_j - \bar{Y})^2, \quad V/\sigma_2^2 \sim \chi^2(n_2 - 1).$

- 检验统计量: 在 $H_0: \sigma_1^2 = \sigma_2^2$ 下,

$$\frac{U/(n_1 - 1)}{V/(n_2 - 1)}$$

- 否定域形式:

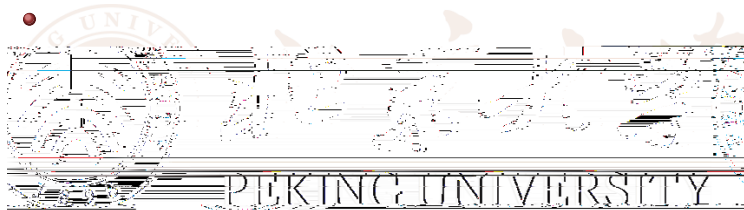
$$\mathcal{W} = \{(\vec{x}, \vec{y}) : \star < F_{\alpha/2}(n_1 - 1, n_2 - 1)$$

$$\text{或 } \star > F_{1-\alpha/2}(n_1 - 1, n_2 - 1)\}.$$

例4.4. 断裂强度试验表(见教材, $n_1 = n_2 = 8$). 比较 σ_1^2 与 σ_2^2 .

- 检验统计量:

$$\frac{U/(n_1 - 1)}{V/(n_2 - 1)} = \frac{\frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (x_i - \bar{x})^2}{\frac{1}{n_2 - 1} \sum_{i=1}^{n_2} (y_i - \bar{y})^2} = 0.9355.$$



B. 均值检验. 假设 $\sigma_1^2 = \sigma_2^2 =: \sigma^2$, 但 σ^2 未知. 检验 $H_0: \mu_1 = \mu_2$.

- $\bar{X} - \bar{Y} \sim N(\mu_1 - \mu_2, \frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2})$.

- 令 $S^2 = \sum_{i=1}^{n_1} (X_i - \bar{X})^2 + \sum_{i=1}^{n_2} (Y_i - \bar{Y})^2$,

则 $S^2/\sigma^2 \sim \chi^2_{n_1 + n_2 - 2}$.

• 检验统计量:

$$T = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \sqrt{\frac{S^2}{n_1 + n_2 - 2}}}$$

• 定理 1. 在 H_0 下, $\mu_1 = \mu_2 = 0$, 于是 $T \sim t(n_1 + n_2 - 2)$.

• 否定域:

$$W = \{(\vec{x}, \vec{y}) : |T| > t_{1-\alpha/2}(n_1 + n_2 - 2)\}.$$

例4.5. 两组病人的胆固醇水平. $n_1 = n_2 = 32$, $\bar{x} = 241.76$ (服药后), $\bar{y} = 224.62$, $s_1 = 51.2808$, $s_2 = 36.2710$.

- $H_0 : \sigma_1^2 \leq \sigma_2^2$, $H_0 : \sigma_1^2 \geq \sigma_2^2$ 都接受:

$$F_{0.025}(31, 31) = 0.4881 < \frac{u/(n_1 - 1)}{v/(n_2 - 1)} = 1.9927$$

$$< F_{0.975}(31, 31) = 2.0486.$$

- 假设 $\sigma_1^2 = \sigma_2^2 = \sigma^2$. 检验 $H_0 : \mu_1 \leq \mu_2$. 否定域:

$$W = \{(\bar{x}, \bar{y}) : T(\bar{x}, \bar{y}) > t_{1-\alpha}(n_1 + n_2 - 2)\}.$$

- 若 $\alpha = 0.1$, 则否定 H_0 ; 若 $\alpha = 0.05$, 则不能否定 H_0 .

$$t_{0.9}(62) < 1.3 < T(\bar{x}, \bar{y}) = 1.5436 < 1.67 \approx t_{0.95}(62).$$

5. 检验的 p 值

- 例4.5, 否定域:

$$W_\alpha = \{(\vec{x}, \vec{y}) : T(\vec{x}, \vec{y}) > t_{1-\alpha}(n_1 + n_2 - 2)\}$$

- α 越小, W_α 就越小.

- 定义4.8: 给定样本来自 $N(\mu, \sigma^2)$, 满足 $\vec{x} \in W_\alpha$ 的最小的 α 称为检验的 p 值, 记为 $p(x_1, \dots, x_n)$.

- p 值越小越好.