1. 广义似然比检验的思想

§8.4 广义似然比检验和关于正态总体参数的检验

- 假设检验问题$H_0 : \theta \in \Theta_0 \leftrightarrow H_1 : \theta \in \Theta_1$.
- 考虑θ 分别在$\Theta = \Theta_0 \cup \Theta_1$ 与Θ_0 中的最大似然估计$\hat{\theta} \in \Theta$ 与$\hat{\theta}_0 \in \Theta_0$:

$$L(\vec{x}, \hat{\theta}) = \sup_{\theta \in \Theta} L(\vec{x}, \theta), \quad L(\vec{x}, \hat{\theta}_0) = \sup_{\theta \in \Theta_0} L(\vec{x}, \theta).$$

- 定义4.1. 称$\lambda(\vec{x}) := L(\vec{x}, \hat{\theta}) / L(\vec{x}, \hat{\theta}_0)$ 为广义似然比.
- 广义似然比否定域指

$$\mathcal{W} := \left\{ \vec{x} : \frac{L(\vec{x}, \hat{\theta})}{L(\vec{x}, \hat{\theta}_0)} > c \right\} = \{\vec{x} : \lambda(\vec{x}) > c\},$$

其中$c \geqslant 1$, 且满足$\sup_{\theta \in \Theta_0} P_\theta(\vec{X} \in \mathcal{W}) = \alpha$.

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A. 单边问题$H_0 : \mu \leqslant \mu_0 \leftrightarrow H_1 : \mu > \mu_0$ (续).

- 广义似然比: $\lambda(\vec{x}) = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}\right)^{\frac{n}{2}}$, 其中,

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2, \quad \hat{\sigma}_0^2 = \begin{cases} \hat{\sigma}^2, & \text{若}\bar{x} \leqslant \mu_0, \\ \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_0)^2, & \text{若}\bar{x} > \mu_0, \end{cases}$$

- 广义似然比否定域: $c_1 \geqslant 1$,

$$W = \left\{\vec{x} : \frac{\hat{\sigma}_0^2}{\hat{\sigma}^2} > c_1\right\} = \left\{\vec{x} : \bar{x} > \mu_0 \text{ 且 } \frac{\sum_{i=1}^{n}(x_i - \mu_0)^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} > c_1\right\}.$$

- $\sum_{i=1}^{n}(x_i - \mu_0)^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\mu_0 - \bar{x})^2$, 因此

$$\frac{\sum_{i=1}^{n}(x_i - \mu_0)^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = 1 + \frac{T^2}{n-1}, \quad \text{其中 } T = \frac{\sqrt{n}(\bar{x} - \mu_0)}{S}.$$

A. 单边问题$H_0 : \mu \leqslant \mu_0 \leftrightarrow H_1 : \mu > \mu_0$ (续).

- 广义似然比: $\mathcal{W} = \left\{\vec{x} : \bar{x} > \mu_0 \text{ 且 } \frac{\sum_{i=1}^n (x_i - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2} > c_1\right\}$. 其中,

$$\frac{\sum_{i=1}^n (x_i - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2} = 1 + \frac{T^2}{n-1}, \quad 其中 T = \frac{\sqrt{n}(\bar{x} - \mu_0)}{S}.$$

- 总结: $c > 0$,

$$\mathcal{W} = \{\vec{x} : T > 0 \text{ 且 } T^2 > c_2\} = \{\vec{x} : T > c\}.$$

- 根据α 求c:

$\forall \mu \leqslant \mu_0$, $T \leqslant \frac{\sqrt{n}(\bar{x}-\mu)}{S} =: T_{n-1} \sim t(n-1)$, 在$\mu = \mu_0$ 时等号成立. 因此, 取$c = t_{1-\alpha}(n-1)$ 即可满足

$$\max_{\mu \leqslant \mu_0} P_\mu(T > c) = P(T_{n-1} > c) = \alpha.$$

B. 双边问题$H_0: \mu = \mu_0 \leftrightarrow H_1: \mu \neq \mu_0$.

- $\theta = (\mu, \sigma^2)$, $\Theta = (-\infty, \infty) \times (0, \infty)$, $\Theta_0 = \{\mu_0\} \times (0, \infty)$.
- 最大似然估计:

$$\hat{\mu} = \bar{x}, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2, \quad L(\vec{x}, \hat{\theta}) = (2\pi\hat{\sigma}^2)^{-\frac{n}{2}} e^{-\frac{n}{2}}$$

$$\hat{\mu} = \mu_0, \quad \hat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_0)^2, \quad L(\vec{x}, \hat{\theta}_0) = (2\pi\hat{\sigma}_0^2)^{-\frac{n}{2}} e^{-\frac{n}{2}}.$$

- 广义似然比否定域: 记$T = \frac{\sqrt{n}(\bar{x}-\mu_0)}{S}$, 则

$$\mathcal{W} = \left\{\vec{x} : \frac{\hat{\sigma}_0^2}{\hat{\sigma}^2} > \tilde{c}\right\} = \{\vec{x} : |T| > c\}.$$

- 根据α 求c: 取$c = t_{1-\alpha/2}(n-1)$ 即可满足

$$P_{\mu_0}(\vec{X} \in \mathcal{W}) = P_{\mu_0}(|T| > c) = \alpha.$$

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A. 双边问题$H_0: \sigma^2 = \sigma_0^2 \leftrightarrow H_1: \sigma^2 \neq \sigma_0^2$ (续).

- 广义似然比:

$$\lambda(\vec{x}) = u^{-\frac{n}{2}} e^{\frac{u}{2}} \left(\frac{e}{n}\right)^{-\frac{n}{2}}, \quad \text{其中 } u = u(\vec{x}) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\sigma_0^2}.$$

- $f(u) := \left(\frac{u}{n}\right)^{-\frac{n}{2}} e^{\frac{u}{2}}$ 关于u 先↓后↑, (导函数先负后正).
- 若$\sigma^2 = \sigma_0^2$, 则$U = U(\vec{X}) = \frac{n\hat{\sigma}^2}{\sigma_0^2} \sim \chi^2(n-1)$.
- 广义似然比否定域:

$$\{\vec{x} : f(u(\vec{x})) > c\} = \{\vec{x} : u(\vec{x}) < c_1 \text{ 或 } u(\vec{x}) > c_2\},$$

其中, c_1, c_2 满足$f(c_1) = f(c_2) = c$ 且$c_1 < c_2$.

A. 双边问题$H_0: \sigma^2 = \sigma_0^2 \leftrightarrow H_1: \sigma^2 \neq \sigma_0^2$ (续).

- UMPU 否定域: 令$g(u) := \left(\frac{u}{n}\right)^{-\frac{n-1}{2}} e^{\frac{u}{2}}$,

$$\{\vec{x} : g(u(\vec{x})) > c\} = \{\vec{x} : u(\vec{x}) < c_3 \text{ 或 } u(\vec{x}) > c_4\},$$

  其中, c_3, c_4 满足$g(c_3) = g(c_4) = c$ 且$c_3 < c_4$,

$$u(\vec{x}) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\sigma_0^2}.$$

- 根据α 求c: 在H_0 下, $U = u(\vec{X}) \sim \chi^2(n-1)$.

  找c 使得$P_{\sigma_0^2}(U < c_3) + P_{\sigma_0^2}(U > c_4) = \alpha$.

- 实际操作: 取$c_5 = \chi^2_{\alpha/2}(n-1)$, $c_6 = \chi^2$

B.

B. 单边问题$H_0: \sigma^2 \geqslant \sigma_0^2 \leftrightarrow H_1: \sigma^2 < \sigma_0^2$ (续).

- 广义似然比否定域:

$$\mathcal{W} = \{\vec{x} : \hat{\sigma}^2 \leqslant \sigma_0^2, \left(\frac{u}{n}\right)^{-\frac{n}{2}} e^{\frac{u}{2}} > \tilde{c}\} = \{\vec{x} : u < c\}.$$

其中$u = u(\vec{x}) = \frac{n\hat{\sigma}^2}{\sigma_0^2}$, $c < n$.

- 根据α 求c.

$\forall \sigma^2 \geqslant \sigma_0^2$, $U := u(\vec{X}) \geqslant \frac{n\hat{\sigma}^2}{\sigma^2} =: U_{n-1} \sim \chi^2(n-1)$,

在$\sigma^2 = \sigma_0^2$ 时, 等号成立.

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A. 方差检验. $H_0 : \sigma_1^2 = \sigma_2^2 \leftrightarrow H_1 : \sigma_1^2 \neq \sigma_2^2$ (续).

- 在Θ_0 中的最大似然估计:
  - 似然函数$L(\vec{x}, \vec{y}, \mu_1, \mu_2, \sigma^2)$:

$$\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{n_1+n_2} \exp\left\{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n_1}(x_i-\mu_1)^2+\sum_{i=1}^{n_2}(y_i-\mu_2)^2\right)\right\}.$$

  - 似然估计:

$$\hat{\mu}_1 = \bar{x}, \quad \hat{\mu}_2 = \bar{y}, \quad \hat{\sigma}^2 = \frac{\sum_{i=1}^{n_1}(x_i-\bar{x})^2 + \sum_{i=1}^{n_2}(y_i-\bar{y})^2}{n_1+n_2}.$$

  - 将似然估计代入似然函数:

$$\hat{L}_0 = \left(\frac{1}{\sqrt{2\pi}}\right)^{n_1+n_2}\left(\frac{1}{\frac{1}{n_1+n_2}(u+v)}\right)^{\frac{n_1+n_2}{2}} e^{-\frac{n_1+n_2}{2}}.$$

A. 方差检验. $H_0: \sigma_1^2 = \sigma_2^2 \leftrightarrow H_1: \sigma_1^2 \neq \sigma_2^2$ (续).

- 广义似然比: $u = \sum_{i=1}^{n_1}(x_i - \bar{x})^2$, $v = \sum_{i=1}^{n_2}(y_i - \bar{y})^2$,

$$\lambda(\vec{x}, \vec{y}) = \frac{\left(\frac{1}{n_1+n_2}(u+v)\right)^{\frac{n_1+n_2}{2}}}{\left(\frac{1}{n_1}u\right)^{\frac{n_1}{2}}\left(\frac{1}{n_2}v\right)^{\frac{n_2}{2}}} = c_0\left(\frac{u}{u+v}\right)^{-\frac{n_1}{2}}\left(\frac{v}{u+v}\right)^{-\frac{n_2}{2}}.$$

- 广义似然比否定域形如

$$\left\{(\vec{x}, \vec{y}) : \frac{u}{u+v} < c_1 \text{ 或 } > c_2\right\}.$$

其中, c_1, c_2 基本上无法计算.

- 实际操作: 取c_3, c_4 使得

$$P_{H_0}\left(\frac{U}{U+V} < c_3\right) = P_{H_0}\left(\frac{U}{U+V} > c_4\right) = \frac{\alpha}{2}.$$

A. 方差检验. $H_0:\sigma_1^2=\sigma_2^2 \leftrightarrow H_1:\sigma_1^2\neq\sigma_2^2$ (续).

- 定义4.2 (F 分布). $F(m_1,m_2)$ 指 $\frac{K_{m_1}/m_1}{K_{m_2}/m_2}$ 的分布, 其中, $K_{m_1}\sim\chi^2(m_1)$, $\tilde{K}_{m_2}\sim\chi^2(m_2)$, 且$K_{m_1},K_{m_2}$ 相互独立. (密度可根据定理3.4.2计算得到.)
- $U=\sum_{i=1}^{n_1}(X_i-\bar{X})^2,\quad U/\sigma_1^2\sim\chi^2(n_1-1)$,
  $V=\sum_{i=1}^{n_2}(Y_i-\bar{Y})^2,\quad V/\sigma_2^2\sim\chi^2(n_2-1)$.
- 检验统计量: 在$H_0:\sigma_1^2=\sigma_2^2$ 下,

$$\frac{U/(n_1-1)}{V/(n_2-1)}\sim F(n_1-1,n_2-1).$$

- 否定域形式:

$$\mathcal{W}=\{(\vec{x},\vec{y}):\star<F_{\alpha/2}(n_1-1,n_2-1)$$
$$\text{或 }\star>F_{1-\alpha/2}(n_1-1,n_2-1)\}.$$

例4.4. 断裂强度试验表(见教材, $n_1 = n_2 = 8$). 比较σ_1^2 与σ_2^2.

- 检验统计量:

$$\frac{U/(n_1-1)}{V/(n_2-1)} = \frac{\frac{1}{n_1-1}\sum_{i=1}^{n_1}(x_i-\bar{x})^2}{\frac{1}{n_2-1}\sum_{i=1}^{n_2}(y_i-\bar{y})^2} = 0.9355.$$

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B. 均值检验. 假设$\sigma_1^2 = \sigma_2^2 =: \sigma^2$, 但$\sigma^2$ 未知. 检验$H_0 : \mu_1 = \mu_2$.

- $\bar{X} - \bar{Y} \sim N(\mu_1 - \mu_2, \frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2})$.
- 令$S^2 = \sum_{i=1}^{n_1}(X_i - \bar{X})^2 + \sum_{i=1}^{n_2}(Y_i - \bar{Y})^2$,
  则$S^2/\sigma^2 \sim \chi^2 n_1 + n_2 - 2$.
- 检验统计量:
$$T = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\sqrt{\frac{S^2}{n_1+n_2-2}}}.$$
- 定理4.1. 在H_0下, $\mu_1 - \mu_2 = 0$, 于是, $T \sim t(n_1 + n_2 - 2)$.
- 否定域:
$$\mathcal{W} = \{(\vec{x}, \vec{y}) : |T| > t_{1-\alpha/2}(n_1 + n_2 - 2)\}.$$

例4.5. 两组病人的胆固醇水平. $n_1 = n_2 = 32$, $\bar{x} = 241.76$(服药后), $\bar{y} = 224.62$, $s_1 = 51.2808$, $s_2 = 36.2710$.

- $H_0 : \sigma_1^2 \leqslant \sigma_2^2$, $H_0 : \sigma_1^2 \geqslant \sigma_2^2$ 都接受:

$$F_{0.025}(31,31) = 0.4881 < \frac{u/(n_1-1)}{v/(n_2-1)} = 1.9927$$

$$< F_{0.975}(31,31) = 2.0486.$$

- 假设$\sigma_1^2 = \sigma_2^2 = \sigma^2$. 检验$H_0 : \mu_1 \leqslant \mu_2$. 否定域:

$$\mathcal{W} = \{(\vec{x},\vec{y}) : T(\vec{x},\vec{y}) > t_{1-\alpha}(n_1+n_2-2)\}.$$

- 若$\alpha = 0.1$, 则否定H_0; 若$\alpha = 0.05$, 则不能否定H_0.

$$t_{0.9}(62) < 1.3 < T(\vec{x},\vec{y}) = 1.5436 < 1.67 \approx t_{0.95}(62).$$

5. 检验的p 值

- 例4.5, 否定域:

$$W_\alpha = \{(\vec{x}, \vec{y}) : T(\vec{x}, \vec{y}) > t_{1-\alpha}(n_1 + n_2 - 2)\}.$$

- α 越小, W_α 就越小.
- 定义4.3. 给定样本$\vec{x} = (x_1, \cdots, x_n)$. 满足$\vec{x} \in W_\alpha$ 的最小的α 称为检验的p 值, 记为$p(x_1, \cdots, x_n)$.
- p 值越小越好.