

§7.6 估计的渐近分布

- 定义6.2. 设 $T_n = T_n(X_1, \dots, X_n)$ 满足:

$$\sqrt{n}(T_n - g(\theta)) \stackrel{d}{\rightarrow} N(0, \sigma^2), \quad \theta \in \Theta,$$

则称 T_n 是渐近正态的, 其中 $\sigma^2 = \frac{2}{\theta}$ 称为渐进方差.

- 工具: CLT & Δ 方法.

- 定理6.3 (Δ 方法). 设 $\sqrt{n}(T_n - g(\theta)) \stackrel{d}{\rightarrow} N(0, \sigma^2)$, $h(\cdot)$ 存在且不为0, 则

$$\sqrt{n}(h(T_n) - h(\theta)) \stackrel{d}{\rightarrow} N(0, h'(\theta)^2 \sigma^2).$$

例6.1. 总体: $X \sim N(\mu, \sigma^2)$, 样本量: n .

- UMVU 估计: $\hat{\mu} = \bar{X}$, $\hat{\sigma}^2 = S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$.
- $\hat{\mu}$ 渐近正态: 事实上,

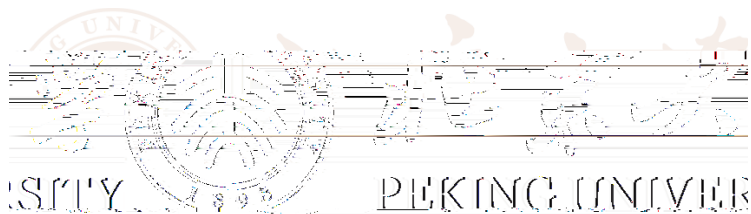
$$\sqrt{n}(\bar{X} - \mu) \xrightarrow{d} N(0, \sigma^2).$$

- 定理7.1. $(n-1)S^2 \xrightarrow{d} \chi^2_{n-1}$, 其中, $\chi^2_{n-1} \sim N(0, 2(n-1))$.

- S^2 渐近正态: CLT,

$$\frac{\sum_{i=1}^{n-1} Z_i^2 - (n-1)}{n-1} \xrightarrow{d} W \sim N(0, \text{var}(Z^2))$$

$$\sqrt{n}(S^2 - \sigma^2) \xrightarrow{d} \sqrt{n-1}(S^2 - \sigma^2) \xrightarrow{d} \sqrt{2}W \sim N(0, 2\sigma^4).$$



例6.3 (续). 总体: $X \sim N(\mu, 1)$, 待估量: $g(\mu) = \Phi(x_0 - \mu)$.

- $\hat{\mu} = \bar{X}$ 是完全充分统计量, 但 $g(\hat{\mu})$ 不是 $g(\mu)$ 的无偏估计.
- 令 $h(\mu) = \Phi\left(\sqrt{\frac{n}{n-1}}(x_0 - \mu)\right)$.
- 记 $p_{\mu, \sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, 则

$$\begin{aligned} E_{\mu} h(\hat{\mu}) &= E_{\mu} \Phi\left(\sqrt{\frac{n}{n-1}}(x_0 - \bar{X})\right) \\ &= \int_{-\infty}^{\infty} p_{\mu, \frac{1}{n}}(y) \Phi\left(\sqrt{\frac{n}{n-1}}(x_0 - y)\right) dy \\ &= \int_{-\infty}^{\infty} p_{\mu, \frac{1}{n}}(y) \int_{-\infty}^{\frac{n}{n-1}(x_0 - y)} p_{0,1}(z) dz dy \\ &= P\left(Z \leq \sqrt{\frac{n}{n-1}}(x_0 - Y)\right), \end{aligned}$$

其中, Y, Z 相互独立, $Y \sim N(\mu, \frac{1}{n})$, $Z \sim N(0, 1)$.

例6.3 (续). 总体: $X \sim N(\mu, 1)$, 待估量: $g(\mu) = \Phi(x_0 - \mu)$.

- 已有: 取 Y, Z 相互独立, $Y \sim N(\mu, \frac{1}{n})$, $Z \sim N(0, 1)$, 则

$$E_{\mu} h(\hat{\mu}) = P\left(Z \leq \sqrt{\frac{n}{n-1}}(x_0 - Y)\right),$$

• $\sqrt{\frac{n-1}{n}}Z = Y \sim N(\mu, 1)$. 因此,

$$E_{\mu} h(\hat{\mu}) = P\left(\sqrt{\frac{n-1}{n}}Z \leq \sqrt{\frac{n-1}{n}}(x_0 - Y) \mid Y \sim N(\mu, 1)\right) = \Phi(x_0 - \mu).$$

- $\hat{\mu}$ 是完全充分统计量, $h(\hat{\mu})$ 是 $g(\mu)$ 的无偏估计, 因此是UMVU 估计.