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例7.3. 假设X 与Y 相互独立, $X\sim\mathcal{P}(\lambda_1)$, $Y\sim\mathcal{P}(\lambda_2)$.
令$Z=X+Y$. 求: Z 的分布; 在$Z=n$ 的条件下, X 的条件分布.

- $Z\sim\mathcal{P}(\lambda_1+\lambda_2)$: $\forall n\geqslant 0$, $\quad P(Z=n)$

$$=\sum_{k=0}^{n}P(X=k,Y=n-k)=\sum_{k=0}^{n}\frac{\lambda_1^k}{k!}e^{-\lambda_1}\frac{\lambda_2^{n-k}}{(n-k)!}e^{-\lambda_2}$$

$$=\frac{1}{n!}\left(\sum_{k=0}^{n}C_n^k\lambda_1^k\lambda_2^{n-k}\right)e^{-(\lambda_1+\lambda_2)}=\frac{(\lambda_1+\lambda_2)^n}{n!}e^{-(\lambda_1+\lambda_2)}.$$

- $n=0$ 时, $P(X=0|Z=0)=1$.
- $n>0$ 时, 记$p=\frac{\lambda_1}{\lambda_1+\lambda_2}$, $q=1-p$. 则, $k=0,1,\cdots n$,

$$P(X=k|Z=n)=\frac{C_n^k\lambda_1^k\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n}=C_n^kp^kq^{n-k},$$

- 条件分布是二项分布.

例7.4. 假设X 与Y 相互独立, $X \sim B(n_1,p)$, $Y \sim B(n_2,p)$, $0<p<1$. 令$Z=X+Y$. 求: Z 的分布; 在$Z=n$ 的条件下, X 的条件分布.

- $n=0,1\cdots,n_1+n_2$. 记$q=1-p$, 则$P(Z=n)$

$$=\sum_{k=0}^{n} P(X=k,Y=n-k)=\sum_{k=0}^{n} C_{n_1}^{k} C_{n_2}^{n-k} p^{k+n-k} q^{n_1-k+n_2-(n-k)}$$

$$=\sum_{k=0}^{n} C_{n_1}^{k} C_{n_2}^{n-k} p^{n} q^{n_1+n_2-n}=C_{n_1+n_2}^{n} p^{n} q^{n_1+n_2-n}.$$

- $n=0$ 时, $P(X=0|Z=0)=1$.
- $n>0$ 时, $k=0,1,\cdots n$,

$$P(X=k|Z=n)=\frac{C_{n_1}^{k} C_{n_2}^{n-k}}{C_{n_1+n_2}^{n}},$$

- 条件分布是超几何分布.

- 定理4.1. 设$\xi = (X, Y)$有联合密度$p(x, y)$, $Z = X + Y$. 则

$$p_Z(z) = \int_{-\infty}^{\infty} p(x, z - x)dx.$$

- 证: 第一步,

$$F_Z(z) = P(X + Y \leqslant z) = \iint_{x+y\leqslant z} p(x, y)dxdy.$$

- 第二步,

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{z-x} p(x, y)dy\right)dx = \int_{-\infty}^{z}\int_{-\infty}^{\infty} p(x, u - x)dxdu$$

- 推论(系4.1): 若X, Y相互独立, 分别有密度p_X, p_Y, 则$Z = X + Y$是连续型, 且

$$p_Z(z) = \int_{-\infty}^{\infty} p_X(x)p_Y(z - x)dx = \int_{-\infty}^{\infty} p_X(z - y)p_Y(y)dy.$$

例4.1 & 4.2. 设$(\mathbf{X},\mathbf{Y})$ 服从二维正态分布, 联合密度$\mathbf{p}(\mathbf{x},\mathbf{y})$ 为

$$\mathbf{p}(\mathbf{x},\mathbf{y}) = \hat{\mathbf{C}} \exp\left\{-\frac{\mathbf{u}^2 - 2\boldsymbol{\rho}\mathbf{u}\mathbf{v} + \mathbf{v}^2}{2(1-\boldsymbol{\rho}^2)}\right\}, \quad \left(\mathbf{u} = \frac{\mathbf{x}-\boldsymbol{\mu}_1}{\boldsymbol{\sigma}_1}, \mathbf{v} = \frac{\mathbf{y}-\boldsymbol{\mu}_2}{\boldsymbol{\sigma}_2}\right),$$

其中, $\hat{\mathbf{C}} = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$. 求$\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ 的密度.

- $\mathbf{p}_Z(\mathbf{z}) = \int_{-\infty}^{\infty} \mathbf{p}(\mathbf{x},\mathbf{z}-\mathbf{x})\mathbf{dx}$. 当$\mathbf{y}$ 取$\mathbf{z}-\mathbf{x}$ 时,

$$\mathbf{v} = \frac{\mathbf{y}-\boldsymbol{\mu}_2}{\boldsymbol{\sigma}_2} = \frac{\mathbf{z}-(\boldsymbol{\mu}_1+\boldsymbol{\sigma}_1\mathbf{u})-\boldsymbol{\mu}_2}{\boldsymbol{\sigma}_2} = \mathbf{C} - \frac{\boldsymbol{\sigma}_1}{\boldsymbol{\sigma}_2}\mathbf{u},$$

其中, $\mathbf{C} = (\mathbf{z}-\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)/\boldsymbol{\sigma}_2$.

- 此时, $\mathbf{u}^2 - 2\boldsymbol{\rho}\mathbf{u}\mathbf{v} + \mathbf{v}^2$

$$= \mathbf{u}^2 - 2\boldsymbol{\rho}\mathbf{u}\left(\mathbf{C} - \frac{\boldsymbol{\sigma}_1\mathbf{u}}{\boldsymbol{\sigma}_2}\right) + \left(\mathbf{C} - \frac{\boldsymbol{\sigma}_1\mathbf{u}}{\boldsymbol{\sigma}_2}\right)^2$$

$$= \left(1 + 2\boldsymbol{\rho}\frac{\boldsymbol{\sigma}_1}{\boldsymbol{\sigma}_2} + \left(\frac{\boldsymbol{\sigma}_1}{\boldsymbol{\sigma}_2}\right)^2\right)\mathbf{u}^2 - 2\left(\boldsymbol{\rho} + \frac{\boldsymbol{\sigma}_1}{\boldsymbol{\sigma}_2}\right)\mathbf{C}\mathbf{u} + \mathbf{C}^2.$$

- 目标: 计算$\mathbf{p}_Z(\mathbf{z}) = \int_{-\infty}^{\infty} \mathbf{p}(\mathbf{x}, \mathbf{z}-\mathbf{x})d\mathbf{x}$. 已有:

$$\mathbf{p}(\mathbf{x}, \mathbf{z}-\mathbf{x}) = \hat{\mathbf{C}}\left\{-\frac{\mathbf{A}\mathbf{u}^2 - 2\mathbf{B}\mathbf{u} + \mathbf{C}^2}{2(1-\rho^2)}\right\}, \quad \text{其中}, \mathbf{u} = \frac{\mathbf{x}-\mu_1}{\sigma_1},$$

$$\mathbf{A} = 1 + 2\rho\frac{\sigma_1}{\sigma_2} + \left(\frac{\sigma_1}{\sigma_2}\right)^2, \quad \mathbf{B} = \left(\rho + \frac{\sigma_1}{\sigma_2}\right)\mathbf{C}, \quad \mathbf{C} = \frac{\mathbf{z} - (\mu_1 + \mu_2)}{\sigma_2}.$$

- 配方:

$$\mathbf{A}\mathbf{u}^2 - 2\mathbf{B}\mathbf{u} + \mathbf{C}^2 = \mathbf{A}\left(\mathbf{u} - \frac{\mathbf{B}}{\mathbf{A}}\right)^2 - \left(\frac{\mathbf{B}^2}{\mathbf{A}} - \mathbf{C}^2\right).$$

- 于是, $\mathbf{p}_Z(\mathbf{z})$

$$= \hat{\mathbf{C}} \exp\left\{\frac{\frac{B^2}{A} - \mathbf{C}^2}{2(1-\rho^2)}\right\} \times \int_{-\infty}^{\infty} \exp\left\{-\frac{\mathbf{A}\left(\mathbf{u} - \frac{B}{A}\right)^2}{2(1-\rho^2)}\right\} \sigma_1 d\mathbf{u}$$

$$= \tilde{\mathbf{C}} \exp\left\{\frac{\mathbf{B}^2 - \mathbf{A}\mathbf{C}^2}{2(1-\rho^2)\mathbf{A}}\right\}. \quad \tilde{\mathbf{C}} = \hat{\mathbf{C}}\sigma_1\sqrt{2\pi\frac{1-\rho^2}{\mathbf{A}}} = \frac{1}{\sqrt{2\pi\sigma_2^2\mathbf{A}}}.$$

- 已有：$\mathbf{p}_Z(\mathbf{z}) = \tilde{\mathbf{C}} \exp\left\{\frac{B^2-AC^2}{2(1-\rho^2)A}\right\}$，其中$\tilde{\mathbf{C}}$ 是常数，

$$\mathbf{A} = 1+2\boldsymbol{\rho}\frac{\boldsymbol{\sigma}_1}{\boldsymbol{\sigma}_2}+\left(\frac{\boldsymbol{\sigma}_1}{\boldsymbol{\sigma}_2}\right)^2, \ \mathbf{B} = \left(\boldsymbol{\rho}+\frac{\boldsymbol{\sigma}_1}{\boldsymbol{\sigma}_2}\right)\mathbf{C}, \ \mathbf{C} = \frac{\mathbf{z}-(\boldsymbol{\mu}_1+\boldsymbol{\mu}_2)}{\boldsymbol{\sigma}_2}.$$

- $\mathbf{B}^2 - \mathbf{A}\mathbf{C}^2$

$$= \left(\left(\boldsymbol{\rho}+\frac{\boldsymbol{\sigma}_1}{\boldsymbol{\sigma}_2}\right)^2 - \mathbf{A}\right)\mathbf{C}^2 = (\boldsymbol{\rho}^2-1)\frac{(\mathbf{z}-(\boldsymbol{\mu}_1+\boldsymbol{\mu}_2))^2}{\boldsymbol{\sigma}_2^2}.$$

- 因此，

$$\mathbf{p}_Z(\mathbf{z}) = \tilde{\mathbf{C}}\exp\left\{-\frac{(\mathbf{z}-\boldsymbol{\mu})^2}{2\boldsymbol{\sigma}^2}\right\} = \frac{1}{\sqrt{2\pi}\boldsymbol{\sigma}}\exp\left\{-\frac{(\mathbf{z}-\boldsymbol{\mu})^2}{2\boldsymbol{\sigma}^2}\right\}.$$

其中，$\boldsymbol{\mu} = \boldsymbol{\mu}_1+\boldsymbol{\mu}_2$，$\boldsymbol{\sigma}^2 = \boldsymbol{\sigma}_2^2\mathbf{A} = \boldsymbol{\sigma}_1^2+\boldsymbol{\sigma}_2^2+2\boldsymbol{\rho}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$.

- 特别地，若$\boldsymbol{\rho} = 0$ (即$\mathbf{X}$，$\mathbf{Y}$ 相互独立)，则

$$\mathbf{X}+\mathbf{Y} \sim \mathbf{N}(\boldsymbol{\mu}_1+\boldsymbol{\mu}_2, \boldsymbol{\sigma}_1^2+\boldsymbol{\sigma}_2^2).$$

- 定理4.2. 设$(\mathbf{X},\mathbf{Y})$有联合密度$\mathbf{p}(\mathbf{x},\mathbf{y})$.
  令$\mathbf{Z}=\mathbf{X}/\mathbf{Y}$ (当$\mathbf{Y}=0$时, 规定$\mathbf{Z}=0$). 则$\mathbf{Z}$为连续型, 且

$$\mathbf{p}_Z(\mathbf{z})=\int_{-\infty}^{\infty}|\mathbf{y}|\mathbf{p}(\mathbf{zy},\mathbf{y})\mathbf{dy}.$$

- 证明: 第一步, $\frac{x}{y}\leqslant \mathbf{z}$ 当且仅当 "$\mathbf{y}>0$ 且$\mathbf{x}\leqslant \mathbf{yz}$" 或者 "$\mathbf{y}<0$ 且$\mathbf{x}\geqslant \mathbf{yz}$." 于是,

$$\mathbf{F}_Z(\mathbf{z})=\mathbf{P}(\mathbf{Y}>0,\mathbf{X}\leqslant \mathbf{Y}\mathbf{z})+\mathbf{P}(\mathbf{Y}<0,\mathbf{X}\geqslant \mathbf{Y}\mathbf{z}).$$

- $\mathbf{P}(\mathbf{Y}>0,\mathbf{X}\leqslant \mathbf{Y}\mathbf{z})$

$$=\int_0^{\infty}\int_{-\infty}^{yz}\mathbf{p}(\mathbf{x},\mathbf{y})\mathbf{dxdy}=\int_0^{\infty}\int_{-\infty}^{z}\mathbf{p}(\mathbf{yu},\mathbf{y})\mathbf{ydudy}$$

$$=\int_{-\infty}^{z}\left(\int_0^{\infty}\mathbf{yp}(\mathbf{yu},\mathbf{y})\mathbf{dy}\right)\mathbf{du}.$$

- 类似处理★★, 即可.

例4.4. $\mathbf{X},\mathbf{Y}$ 相互独立, 都服从$\mathbf{N}(0,1)$. 求$\mathbf{Z}=\mathbf{X}/\mathbf{Y}$ 的密度.

- 联合密度:

$$p(x,y)=\frac{1}{2\pi}\exp\left\{-\frac{x^2+y^2}{2}\right\}.$$

- 因此,

$$p_Z(z)=\int_{-\infty}^{\infty}|y|p(zy,y)dy=\int_{-\infty}^{\infty}|y|\frac{1}{2\pi}\exp\left\{-\frac{(zy)^2+y^2}{2}\right\}dy$$

$$=\frac{2}{2\pi}\int_0^{\infty}y\exp\left\{-\frac{(z^2+1)y^2}{2}\right\}dy$$

$$=\frac{1}{\pi}\int_0^{\infty}e^{-(z^2+1)u}du=\frac{1}{\pi(z^2+1)}.$$

- 定理4.3. 假设$\xi = (X, Y)$为连续型, 有密度$p(x, y)$. 假设

$$\eta = (U, V), \qquad \text{其中 } U = f(X, Y), \quad V = g(X, Y).$$

如果(1) $P(\xi \in A) = 1$ 且$(f, g): A \to G$ 是一对一的;

(2) $f, g \in C^1(A)$, 且$\frac{\partial(u,v)}{\partial(x,y)} \neq 0, \forall (x, y) \in A$;

那么, η 是连续型, 且

$$p_{U,V}(u, v) = p\big(x(u, v), y(u, v)\big) \left|\frac{\partial(x, y)}{\partial(u, v)}\right|, \ (u, v) \in G.$$

- 证: $\forall D \subseteq G$, 找$D^* \subseteq A$ 使得$\eta \in D$ iff $\xi \in D^*$. 于是,

$$P(\star) = P(\star) = \iint_{D*} p(x, y)dxdy = \iint_{D} \star\star dudv.$$

例4.5, 4.7, & 习题三、21. 假设$\mathbf{X}, \mathbf{Y}$ 相互独立, 都服从$\mathbf{N}(0,1)$.

- 用极坐标表达:

$$\mathbf{X} = \mathbf{R}\cos\Theta, \quad \mathbf{Y} = \mathbf{R}\sin\Theta.$$

- $\mathbf{A} = \{(\mathbf{x},\mathbf{y}) : \mathbf{x} \neq 0, \mathbf{y} \neq 0\}$,

  $\mathbf{G} = \{(\mathbf{r},\boldsymbol{\theta}) : \mathbf{r} > 0, 0 \le \boldsymbol{\theta} < 2\pi;\ \boldsymbol{\theta} \neq \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$.

- $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\boldsymbol{\theta} & -\mathbf{r}\sin\boldsymbol{\theta} \\ \sin\boldsymbol{\theta} & \mathbf{r}\cos\boldsymbol{\theta} \end{vmatrix} = \mathbf{r}$.

- $\mathbf{p}_{R,\Theta}(\mathbf{r},\boldsymbol{\theta}) = \frac{1}{2\pi}\mathbf{e}^{-\frac{1}{2}(x^2+y^2)}\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|$

  $= \frac{1}{2\pi}\mathbf{r}\mathbf{e}^{-\frac{1}{2}r^2},\ \mathbf{r} > 0,\ 0 < \boldsymbol{\theta} < 2\pi;\ \boldsymbol{\theta} \neq \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

- $\mathbf{R}, \Theta$ 独立: $\mathbf{p}_{R,\Theta}(\mathbf{r},\boldsymbol{\theta}) = \mathbf{p}_R(\mathbf{r}) \cdot \mathbf{p}_\Theta(\boldsymbol{\theta})$.

- $\mathbf{W} := \mathbf{R}^2 = \mathbf{X}^2 + \mathbf{Y}^2 \sim \mathrm{Exp}(\frac{1}{2})$. 因为, $\forall \mathbf{w} > 0$,

$$p_W(\mathbf{w}) = p_R(\mathbf{r}) \frac{d\mathbf{r}}{d\mathbf{w}} = \mathbf{r} \exp\left\{ -\frac{\mathbf{r}^2}{2} \right\} \cdot \frac{1}{2\mathbf{r}} = \frac{1}{2} e^{-\frac{w}{2}}.$$

- $\mathbf{U} := e^{-\frac{1}{2}W} \sim \mathbf{U}(0,1)$. 因为, $\forall \mathbf{p} \in (0,1)$,

$$P(\mathbf{U} < \mathbf{p}) = P(\mathbf{W} > -2\ln \mathbf{p}) = e^{-\frac{1}{2}(-2\ln p)} = e^{\ln p} = \mathbf{p}.$$

- $\mathbf{V} := \frac{1}{2\pi}\Theta \sim \mathbf{U}(0,1)$, 且$\mathbf{U}$与$\mathbf{V}$相互独立
- $\mathbf{R} = \sqrt{-2\ln \mathbf{U}}$, $\Theta = 2\pi\mathbf{V}$, 即

$$\mathbf{X} = \sqrt{-2\ln \mathbf{U}} \cos(2\pi\mathbf{V}), \quad \mathbf{Y} = \sqrt{-2\ln \mathbf{U}} \sin(2\pi\mathbf{V}).$$

2.两个随机变量的函数的数学期望

- 随机向量函数的期望(定理4.6):

  离散型: $\mathbf{E}\,\mathbf{f}(\mathbf{X},\mathbf{Y}) = \sum_{i,j} \mathbf{f}(\mathbf{x}_i,\mathbf{y}_j)\mathbf{P}(\mathbf{X}=\mathbf{x}_i,\mathbf{Y}=\mathbf{y}_j)$.

  连续型: $\mathbf{E}\,\mathbf{f}(\mathbf{X},\mathbf{Y}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{f}(\mathbf{x},\mathbf{y})\mathbf{p}(\mathbf{x},\mathbf{y})\mathbf{dxdy}$.

- 定理4.4. 若X 与Y 相互独立, 则$\mathbf{EXY} = (\mathbf{EX})\cdot(\mathbf{EY})$.
- 例, 连续型的证明:

  $$\mathbf{EXY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{xy}\mathbf{p}_X(\mathbf{x})\mathbf{p}_Y(\mathbf{y})\mathbf{dxdy} = (\mathbf{EX})(\mathbf{EY}).$$

- 定理4.5. 若X 与Y 相互独立,

  则$\mathrm{var}(\mathbf{X}+\mathbf{Y}) = \mathrm{var}(\mathbf{X}) + \mathrm{var}(\mathbf{Y})$.
- 证: 左$= \mathbf{E}(\mathbf{X}+\mathbf{Y}-(\mathbf{EX}+\mathbf{EY}))^2$

  $=$ 右$+2\mathbf{E}(\mathbf{X}-\mathbf{EX})(\mathbf{Y}-\mathbf{EY})$.