

§2.6 随机变量的数学期望

期望(expectation)的含义: 均值(mean).

- X 的大量独立观测值(记为 $a_1 a_2 \cdots a_n$) 的算术平均:

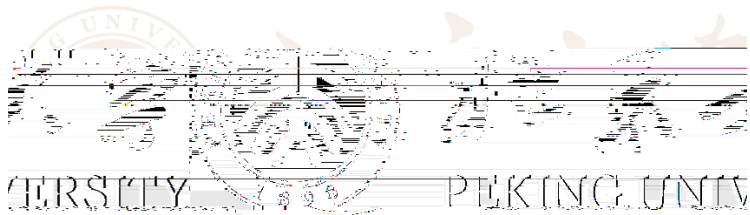
$$\bar{a} = \frac{1}{n}(a_1 + \cdots + a_n).$$

- X 的所有可能值的加权平均(总和)

例, $(X = x_k) = p_k, 1, \dots$

记 $n_k = \{ 1 \leq m \leq n, a_m = x_k \}$. 那么, 根据概率的频率含义, $\frac{n_k}{n} \approx p_k$, 于是

$$\bar{a} = \frac{1}{n} \sum_{k=1}^K n_k \approx \sum_{k=1}^K x_k p_k$$



(2) 二项分布.

$$(X = k) = C_n^k p^k q^{n-k} =: (n; k) \quad k = 0, 1, \dots, n \quad (q = 1 - p),$$

• $\forall 1 \leq k \leq n,$

$$\begin{aligned} (n; k) &= \frac{n!}{k!(n-k)!} p^k q^{n-k} = \frac{n!}{(k-1)!(n-k)!} p^{k-1} q^{n-k} \cdot p \\ &= np \cdot \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} q^{n-k} = np \cdot (n-1; k-1). \end{aligned}$$

$$\begin{aligned} EX &= \sum_{k=0}^n k \cdot (n; k) = \sum_{k=1}^n np \cdot (n-1; k-1) \\ &= np \sum_{\ell=0}^{n-1} (n-1; \ell) = np. \end{aligned}$$

(7) 超几何分布.

$$P(X = k) = \frac{C_D^k C_{N-D}^{n-k}}{C_N^n} \quad k = 0, 1, \dots, n,$$

• 记 $P(X = k) = A_1 \cdot A_2 \cdot A_3 =$

$$\frac{D!}{k!(D-k)!} \cdot \frac{(n-k)!}{(n-k-D)!(n-k)!} \cdot \frac{n!}{k!(n-k)!}$$

• 记 $x' = x - 1$, 则 $\forall 1 \leq x' \leq n$,

$$A_1 = \frac{D!}{(k-1)!(D-(k-1))!} = D \times \frac{D!}{k!(D-k)!}$$

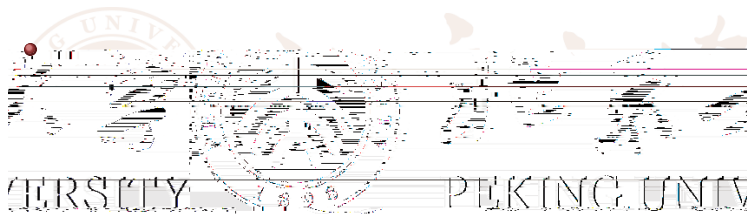
• 进一步,

$$A_2 = \frac{(n-k-D)!}{(n-k-D)!(n-k-D)!(n-k-D)!}$$

$$A_3 = \frac{n \cdot n!}{k!(n-k)!} = \frac{n}{k} \times \frac{n!}{k!(n-k)!}$$

- 记 $x' = x - 1$. 则 $\forall 1 \leq i \leq n$,

$$D_{n, i} = \frac{nD}{i} \times D'_{n', i'}$$



(4) 几何分布.

$$(X = k) = q^{k-1}p =: p_k \quad k = 1, 2, \dots \quad (q = 1 - p).$$

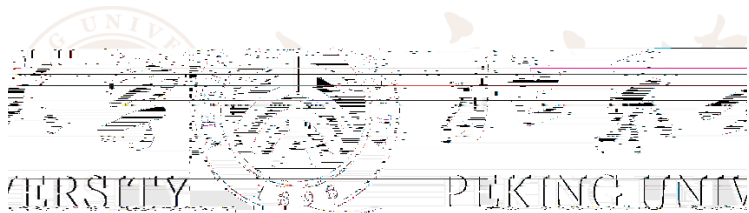
• 直接计算:

$$EX = \sum_{k=1}^{\infty} k p_k = \sum_{k=1}^{\infty} \sum_{\ell=1}^k p_k = \sum_{\ell=1}^{\infty} \sum_{k=\ell}^{\infty} p_k$$

$$= \sum_{\ell=1}^{\infty} p \frac{q^{\ell-1}}{1-q} = \sum_{m=0}^{\infty} \frac{q^m}{1-q} = \frac{1}{1-q} \frac{1}{p}$$

• 习题二、18. 若 X 取非负整数, 则 $EX = \sum_{\ell=1}^{\infty} (X \geq \ell)$.

• 证: $\sum_{k=\ell}^{\infty} p_k = (X \geq \ell)$.



(2) 指数分布.

$$p(x) = \lambda e^{-\lambda x} \quad x > 0,$$

$$\int_0^{\infty} x \cdot \lambda e^{-\lambda x} dx = - \int_0^{\infty} x e^{-\lambda x} dx = \int_0^{\infty} e^{-\lambda x} dx = \frac{1}{\lambda}.$$

(3) 正态分布.

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- $X \sim (0, 1)$:

$$EX = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0$$

- 同理, $X \sim (-\infty, \infty)$, 则 $p(+x) = p(-x)$, 因此 $EX = 0$.

- 例, 柯西分布,

$$p(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$$

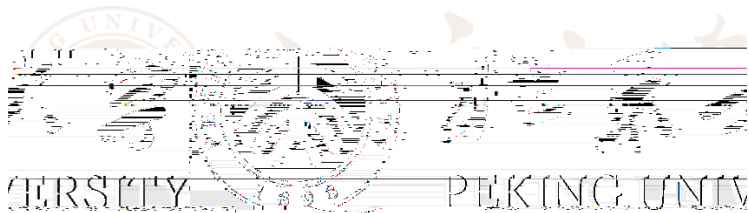
但是, $\int_{-\infty}^{\infty} |x|p(x) dx = \infty$. 因此, **EX 不存在!**

(4) 伽玛分布.

$$p(x) = \frac{\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad x > 0,$$

• $\forall x > 0,$

$$x p(x) = \frac{\alpha}{\Gamma(\alpha)} x^{\alpha} e^{-\beta x} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \cdot \frac{\alpha+1}{\Gamma(\alpha+1)} x^{\alpha} e^{-\beta x} = \hat{p}(x).$$



3. 期望的性质

- 推论6.2. (1) 线性: 假设 EX, EY 存在. 则,

$$E(aX + Y) = aEX + EY,$$

- 推论6.2. (2) 和的期望: 假设 EX_1, \dots, EX_n 都存在,

$X = X_1 + \dots + X_n$. 则 E 存在, 且

$$E = EX_1 + \dots + EX_n.$$

- 例. 超几何分布 $\sim H(D, n)$.

若第 i 个产品是次品, 则令 $X_i = 1$; 否则, 令 $X_i = 0$. 则,

$$X = X_1 + \dots + X_n \Rightarrow E = np.$$

- 定理6.4. (马尔可夫不等式). 设 $X \geq 0$, 且 EX 存在. 则对任意 $C > 0$, 有

$$(X \geq C) \leq \frac{1}{C} EX,$$

- 证: 令 $A = \{X \geq C\}$. 则 $1_A \leq \frac{X}{C}$. 于是,

$$(A) \quad E1_A \leq E \frac{X}{C} = \frac{1}{C} EX$$

- 例, 若 $X \geq 0$, 且 $EX = 0$, 则

$$\left(X \geq \frac{1}{n}\right) \leq nEX = 0$$

$$\Rightarrow (X > 0) = \lim_{n \rightarrow \infty} \left(X \geq \frac{1}{n}\right) = 0,$$

