§2.6 随机变量的数学期望

期望(expectation)的含义: 均值(mean).

- X 的大量独立观测值(记为$a_1 \ a_2 \ \cdots \ a_n$) 的算术平均:

$$\bar{a} = \frac{1}{n}(a_1 + \cdots + a_n).$$

- X 的所有可能值的加权平均(总和).

例, $(X = x_k) = p_k, \ 1 \cdots.$

记$n_k = \{ \ : 1 \leq \ \leq n \ a_m = x_k\}$. 那么, 根据概率的频率含义, $\frac{n_k}{n} \approx p_k$, 于是

$$\bar{a} = \frac{1}{n}\sum_{k=1}^{K} n_k \approx \sum_{k=1}^{K} x_k p_k$$

VERSITY PEKING UNIV

(2) 二项分布.

$$(X = \quad) = C_n^k p^k q^{n-k} =: \quad (n; \quad) \qquad = 0 \ 1 \ \cdots \ n \ (q = 1 - p),$$

- $\forall 1 \leqslant \quad \leqslant n,$

$$\cdot \ (n; \quad) = \quad \cdot \frac{n!}{!(n - \quad)!} p^k q^{n-k} = \frac{n!}{(\quad - 1)!(n - \quad)!} p^k q^{n-k}$$

$$= \frac{n \cdot (n-1)!}{(\quad - 1)!(n - \quad)!} p \cdot p^{k-1} q^{n-k} = np \cdot \quad (n - 1; \quad - 1).$$

- 因此,

$$EX = \sum_{k=0}^{n} \quad \cdot \quad (n; \quad) = \sum_{k=1}^{n} np \cdot \quad (n-1; \quad - 1)$$

$$= np \sum_{\ell=0}^{n-1} \quad (n-1 \quad) = np.$$

(7) 超几何分布.

$$(X = \quad) = \frac{C_D^k C_{N-D}^{n-k}}{C_N^n} \qquad = 0\ 1\ \cdots\ n,$$

- 记, $(\quad D\ n;\ ) = A_1 \cdot A_2 \cdot A_3 =$

$$\frac{D!}{!(D - \ )!} \cdot \frac{(\quad - D)!}{(n - \ )!(\quad - D - (n - \ ))!} \cdot \frac{n!(\quad - n)!}{!},$$

- 记$x' = x - 1$, 则, $\forall 1 \leqslant \ \leqslant n$,

$$A_1 = \frac{D!}{(\ - 1)!(D - \ )!} = D \times \frac{D'!}{'!(D' - \ ')!}$$

- 进一步,

$$A_2 = \frac{(\ ' - D')!}{(n' - \ ')!(\ ' - D' - (n' - \ '))!}$$

$$A_3 = \frac{n \cdot n'!(\ ' - n')!}{\cdot \ '!} = \frac{n}{\ } \times \frac{n'!(\ ' - n')!}{'!},$$

- 记$x' = x - 1$. 则$\forall 1 \leqslant \quad \leqslant n$,

$$\cdot, (\quad D \; n; \;) = \frac{nD}{} \times, (\quad ' \; D' \; n'; \; ')$$

-

(4) 几何分布.

$$(X = \quad) = q^{k-1}p =: p_k \qquad = 1 \ 2 \ \cdots \quad (q = 1-p)$$

- 直接计算:

$$EX = \sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} \sum_{\ell=1}^{k} p_k = \sum_{\ell=1}^{\infty} \sum_{k=\ell}^{\infty} p_k$$

$$= \sum_{\ell=1}^{\infty} p \cdot \frac{q^{\ell-1}}{1-q} = \sum_{m=0}^{\infty} q^m = \frac{1}{1-q} = \frac{1}{p}$$

- 习题二、18. 若X 取非负整数, 则$EX = \sum_{\ell=1}^{\infty} \quad (X \geqslant \ )$.
- 证: $\sum_{k=\ell}^{\infty} p_k = \quad (X \geqslant \ )$.

ERSITY PEKING UNIV

(2) 指数分布.

$$p(x) = \lambda e^{-\lambda x} \qquad x > 0,$$

- $\int_0^\infty x \cdot \lambda e^{-\lambda x} dx = -\int_0^\infty x d e^{-\lambda x} = \int_0^\infty e^{-\lambda x} dx = \frac{1}{\lambda}.$
-

(3) 正态分布.

$$p(x) = \frac{1}{\sqrt{2}} \; ^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

- $X \sim \quad (0 \; 1)$:

$$EX = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2}} \; ^{-\frac{x^2}{2}} \; x = 0.$$

- 同理, $X \sim \quad (\quad {}^2)$, 则 $p(\quad + x) = p(\quad - x)$, 因此 $EX = \quad$.
- 例, 柯西分布,

$$p(x) = \frac{1}{\;} \cdot \frac{1}{1+x^2},$$

但是, $\int_{-\infty}^{\infty} |x| p(x) \; x = \infty$. 因此, EX 不存在!

(4) 伽玛分布.

$$p(x) = \frac{\alpha}{\Gamma(\ )} x^{\alpha-1} \ ^{-\beta x} \qquad x > 0$$

- $\forall x > 0,$

$$xp(x) = \frac{\alpha}{\Gamma(\ )} x^{\alpha} \ ^{-\beta x} = \frac{\Gamma(\ +1)}{\Gamma(\ )} \cdot \frac{\alpha+1}{\Gamma(\ +1)} x^{\alpha} \ ^{-\beta x} = \frac{}{} \cdot \hat{p}(x),$$

- 因

ERSITY PEKING UNIV

- 推论6.2. (1) 线性: 假设EX, EY 存在. 则,

$$E(aX + \ Y) = aEX + \ EY$$

- 推论6.2. (2) 和的期望: 假设$EX_1, \cdots, EX_n$ 都存在,

  $= X_1 + \cdots + X_n$. 则E 存在, 且

$$E \ = EX_1 + \cdots + EX_n$$

- 例. 超几何分布 $\sim H(\ \ D \ n)$.

  若第 个产品是次品, 则令$X_i = 1$; 否则, 令$X_i = 0$. 则,

$$= X_1 + \cdots + X_n \Rightarrow E \ = np$$

- 定理6.4. (马尔可夫不等式). 设$X \geqslant 0$, 且EX 存在. 则对任意$C > 0$, 有

$$(X \geqslant C) \leqslant \frac{1}{C} EX,$$

- 证: 令$A = \{X \geqslant C\}$. 则$1_A \leqslant \frac{X}{C}$. 于是,

$$(A) = E1_A \leqslant E\frac{X}{C} = \frac{1}{C} EX.$$

- 例, 若$X \geqslant 0$, 且$EX = 0$, 则

$$\left(X \geqslant \frac{1}{n}\right) \leqslant nEX = 0$$

$$\Rightarrow \quad (X > 0) = \lim_{n\to\infty} \left(X \geqslant \frac{1}{n}\right) = 0.$$

ERSITY PEKING UNIV