Supplementary Material for "Online Estimation for Functional Data"

In this Supplementary Material, we first derive the asymptotic distributions of the proposed mean and covariance estimates, i.e., Theorem 1 and 2, in Section S.1 and S.2, respectively. The convergence of bandwidth selection and lower bound of relative efficiencies are shown in Section S.3 and S.4.

S.1. PROOF OF THEOREM 1

Proof. This proof is in analogy to the proof of Theorem 3.1 in Zhang and Wang (2016). Denote the current time by K and recall $\{\widetilde{h}_{\mu,k}^{(K)}, k = 1, \ldots, K\}$ is the pseudo-bandwidth chain at time K. We omit the superscript "(K)" of $\widetilde{h}_{\mu,k}^{(K)}$, $\hat{\theta}^{(K)}$ in (18), $\gamma_{\mu,j}^{(K)}$ in (14) and $\widetilde{\mu}^{(K)}$ in (10) in the proof when no confusion exits. The online estimate can be written as

$$\widetilde{\mu}(t) = \frac{R_0 Q_2 - R_1 Q_1}{Q_0 Q_2 - Q_1^2}$$

where for $r = 0, 1, 2$,

$$Q_r = \frac{1}{S_{K,1}} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j=1}^{m_{ki}} \widetilde{W}_{\mu,k}(T_{kij} - t) \left(\frac{T_{kij} - t}{\widetilde{h}_{\mu,k}} \right)^r = \sum_k \omega_k Q_{r,k},$$

$$R_r = \frac{1}{S_{K,1}} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j=1}^{m_{ki}} \widetilde{W}_{\mu,k}(T_{kij} - t) \left(\frac{T_{kij} - t}{\widetilde{h}_{\mu,k}} \right)^r Y_{kij} = \sum_k \omega_k R_{r,k}.$$

We further denote for $r = 0, 1$,

$$\widehat{G}_r = \frac{1}{S_{K,1}} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j=1}^{m_{ki}} \widetilde{W}_{\mu,k}(T_{kij} - t) T_{kij}^r = \sum_k \omega_k \widehat{G}_{r,k},$$

and define

$$\widetilde{\mu}'(t) = \frac{1}{\widetilde{h}_{\mu,k}} \frac{R_1 - R_0 Q_1 / Q_0}{Q_2 - Q_1^2 / Q_0}.$$

Note that

$$Q_1 = \sum_k \omega_k Q_{1,k} = \sum_k \omega_k \frac{\widehat{G}_{1,k} - t\widehat{G}_{0,k}}{\widetilde{h}_{\mu,k}},$$

one can derive

$$\widetilde{\mu}(t) = \frac{R_0}{Q_0} - \sum_k \omega_k \frac{\widehat{G}_{1,k}}{Q_0} \widetilde{\mu}'_k(t) + t \sum_k \omega_k \frac{\widehat{G}_{0,k}}{Q_0} \widetilde{\mu}'_k(t).$$

We also de ne

$$^{-}(t) = \frac{R_0}{Q_0} \sum_k !_k \frac{\mathfrak{S}_{1;k}}{Q_0} \; '(t) + t \sum_k !_k \frac{\mathfrak{S}_{0;k}}{Q_0} \; '(t); \qquad \text{(S.1)}$$

where $'(\,)$ is the rst derivative of $(\,)$. Then

$$e(t) \quad ^{-}(t) = \sum_k !_k \frac{Q_{1;k}}{Q_0} \frac{Q_0(R_1 \quad Q_1 \quad e_{\;;k} \; 'Q_2) \quad Q_1(R_0 \quad Q_0 \quad e_{\;;k} \; 'Q_1)}{Q_2 Q_0 \quad Q_1^2} : \qquad \text{(S.2)}$$

It is straightforward to show that both Q_0 and $Q_0 Q_2 \quad Q_1^2$ are positive and bounded away from 0 with probability tending to one,

$$Q_1 = O_p \left(\;_{;1} + \left(\frac{\;_{;-1}}{S_{K;1}} \right)^{\frac{1}{2}} \right);$$

$$(R_1 \quad Q_1 \quad e_k \; 'Q_2); (R_0 \quad Q_0 \quad e_k \; 'Q_1) = O_p \left(\;_{;2} + \;_{;1} e_{\;;k} + \left(\frac{\;_{;-1}}{S_{K;1}} \right)^{\frac{1}{2}} + \left(\frac{S_{K;2}}{S_{K;1}^2} \right)^{\frac{1}{2}} \right):$$

Then by the Cramer-Wold device and Lyapunov condition, we can achieve the asymptotic joint normality of $(R_0 \quad \mathrm{E}R_0; \mathfrak{S}_1 \quad \mathrm{E}\mathfrak{S}_1; \mathfrak{S}_0 \quad \mathrm{E}\mathfrak{S}_0)$, where the rate of convergence is $\min[(\;_{;-1}/S_{K;1})^{1/2}; \{Q_2/S_{K;1}^2\}^{1/2}]$.

Explicitly for $r; r' = 0; 1$,

$$\mathrm{E}(\mathfrak{S}_r) = t^r f(t) + \frac{1}{2} \;(W) \{2 r f'(t) + t^r f''(t)\} \;_{;2} + o_p(\;_{;2});$$

$$\mathrm{E}(R_0) = \;(t) f(t) + \frac{1}{2} \;(W) \{\;(t) f''(t) + 2 \; '(t) f'(t) + \; '' f(t)\} \;_{;2} + o_p(\;_{;2});$$

$$\mathrm{Cov}(\mathfrak{S}_r; \mathfrak{S}_{r'}) = \frac{\;_{;-1}}{S_{K;1}} R(W) t^{r+r'} f(t) + o_p \left(\frac{\;_{;-1}}{S_{K;1}} \right);$$

$$\mathrm{Var}(R_0) = \frac{\;_{;-1}}{S_{K;1}} R(W)(\;^2(t) + \;(t;t) + \;^2) f(t) + \frac{S_{K;2}}{S_{K;1}^2} \;(t;t) f(t)^2 + o_p \left(\frac{\;_{;-1}}{S_{K;1}} + \frac{S_{K;2}}{S_{K;1}^2} \right);$$

$$\mathrm{Cov}(R_0; \mathfrak{S}_r) = \frac{\;_{;-1}}{S_{K;1}} R(W) t^r \;(t) f(t) + o_p \left(\frac{\;_{;-1}}{S_{K;1}} \right):$$

From the delta method, $^{-}(t)$ follows the asymptotic normality that

$$^{-1/2} \left\{ ^{-}(t) \quad (t) \quad \frac{1}{2} \;(W) \; ^{(2)}(t) \;_{;2} + o_p(\;_{;2}) \right\} \xrightarrow{d} N(0; 1);$$

where

$$= R(W) \frac{\;(t;t) + \;^2}{S_{K;1} f(t)} \;_{;-1} + \;(t;t) \frac{S_{K;2}}{S_{K;1}^2}:$$

When $\overline{m}_{\;;K} / (N_K)^{1/4} \; ! \; 0$ and $\mathfrak{h} \quad S_{K;1}^{-1/5}$, the rst term in above dominates. When $\overline{m}_{\;;K} / (N_K)^{1/4} \; ! \; C$ and $\mathfrak{h} \quad S_{K;1}^{-1/5}$, where $0 < C < 1$, the two terms of are of the same order as $\hat{h}^4$. When $\overline{m}_{\cdot;K}=(N_K)^{1/4}\to\infty$ and $\hat{h}\asymp S_{K;1}^{-1/5}$, the second term of dominates and the bias becomes negligible. Noting (S.2), the proof of Theorem 1 is completed. ■

S.2. PROOF OF THEOREM 2 AND LEMMA 1

We omit the superscript "(K)" of $e_{\cdot;k}^{(K)}$, $\hat{h}^{(K)}$, $\gamma_{\cdot;j}^{(K)}$ and $e^{(K)}$ in this section when no confusion exits. The proof of Lemma 1 is similar to the proof of Theorem 2, with $\gamma_{\cdot;j}, j\in\mathbb{Z}$ replaced by $\hat{h}$. Before delineating the proof of Theorem 2, we state the following lemma which is a preparation for the proof of Theorem 2.

Lemma S. 1. *Let*

$$Q_{0R_1} = \frac{1}{S_{K;1}}\sum_{k'=1}^{K}\sum_{i'=1}^{n_{k'}}\sum_{j'=1}^{m_{k'i'}} \tilde{W}_{\cdot;k'}(T_{k'i'j'}-T_{kil});$$

$$\hat{R}_1 = S_{K;1}^{-1}S_{K;2}^{-1}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j\neq l}\sum_{k'=1}^{K}\sum_{i'=1}^{n_{k'}}\sum_{j'=1}^{m_{k'i'}} \tilde{W}_{\cdot;k'}(T_{kij}-s)\tilde{W}_{\cdot;k'}(T_{kil}-t)\tilde{W}_{\cdot;k'}(T_{k'i'j'}-T_{kil}) \left(Y_{kij}-\mu(T_{kij})\right)\left[Y_{k'i'j'}-\mu(T_{k'i'j'})-\frac{1}{2}\mu''(T_{kil})(T_{k'i'j'}-T_{kil})^2\right];$$

then

$$\mathrm{E}(\hat{R}_1/Q_{0R_1}) = O_p(S_{K;1}^{-1}S_{K;2}^{-1}S_{K;3});$$

$$\mathrm{Var}(\hat{R}_1/Q_{0R_1}) = O_p\Big(S_{K;1}^{-2}S_{K;2}^{-2}\Big\{S_{K;3}^2+\sum_k s_{k;3}e_{\cdot;k}^{-1}e_{\cdot;k}^{-2}+\sum_k s_{k;4}e_{\cdot;k}^{-1}e_{\cdot;k}^{-1} + \sum_k s_{k;5}e_{\cdot;k}^{-1}\Big\}+S_{K;2}^{-2}\sum_k s_{k;4}e_{\cdot;k}^4\Big).$$

Proof. By easy calculation,

$$\begin{aligned}
\mathrm{E}\,\hat{R}_1 &= S_{K;1}^{-1}S_{K;2}^{-1}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j\neq l}\sum_{k'=1}^{K}\sum_{i'=1}^{n_{k'}}\sum_{j'=1}^{m_{k'i'}}\mathrm{E}\Big[\tilde{W}_{\cdot;k'}(T_{kij}-s)\tilde{W}_{\cdot;k'}(T_{kil}-t)\\
&\qquad \tilde{W}_{\cdot;k'}(T_{k'i'j'}-T_{kil})(Y_{kij}-\mu(T_{kij}))(Y_{k'i'j'}-\mu(T_{k'i'j'}))\Big]\\
&= S_{K;1}^{-1}S_{K;2}^{-1}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j\neq l}\sum_{i'=1}^{m_{ki}}\mathrm{E}\Big[\tilde{W}_{\cdot;k}(T_{kij}-s)\tilde{W}_{\cdot;k}(T_{kil}-t)\tilde{W}_{\cdot;k}(T_{kij'}-T_{kil})\\
&\qquad (Y_{kij}-\mu(T_{kij}))(Y_{kij'}-\mu(T_{kij'}))\Big]\\
&= S_{K;1}^{-1}S_{K;2}^{-1}\sum_{k=1}^{K}\Big[\sum_{j} m_{ki}^2(m_{ki}-1)\mathrm{E}\,\tilde{W}_{\cdot;k}(T_{kij}-s)\tilde{W}_{\cdot;k}(T_{kil}-t)
\end{aligned}$$

$$\#$$

$$\widetilde{W}_{;k}(T_{kij'} \quad T_{kil})(Y_{kij} \quad (T_{kij}))(Y_{kij'} \quad (T_{kij'}))$$

$$= O_p \quad S_{K;1}^{-1} S_{K;2}^{-1} S_{K;3} \quad ;$$

and hence E $\widehat{R}_1 = Q_{0R_1} = O_p(S_{K;1}^{-1} S_{K;2}^{-1} S_{K;3})$. Note that

$$\widehat{R}_1^2 = S_{K;1}^{-2} S_{K;2}^{-2} \sum_{j_1 \neq l_1; j_2 \neq l_2} W_{k_1 i_1 j_1}(s) W_{k_1 i_1 l_1}(t) W_{k'_1 i'_1 j'_1}(T_{k_1 i_1 l_1}) W_{k_2 i_2 j_2}(s) W_{k_2 i_2 l_2}(t) W_{k'_2 i'_2 j'_2}(T_{k_2 i_2 l_2})$$

$$(Y_{k_1 i_1 j_1} \quad {}_{k_1 i_1 j_1}) \quad Y_{k'_1 i'_1 j'_1} \quad {}_{k'_1 i'_1 j'_1} \quad \frac{1}{2} \quad {}''_{k_1 i_1 l_1}(T_{k'_1 i'_1 j'_1} \quad T_{k_1 i_1 l_1})^2$$

$$(Y_{k_2 i_2 j_2} \quad {}_{k_2 i_2 j_2}) \quad Y_{k'_2 i'_2 j'_2} \quad {}_{k'_2 i'_2 j'_2} \quad \frac{1}{2} \quad {}''_{k_2 i_2 l_2}(T_{k'_2 i'_2 j'_2} \quad T_{k_2 i_2 l_2})^2 \quad ;$$

E $\widehat{R}_1^2$ is the summation of the following quantities:

$$(a) = S_{K;1}^{-2} S_{K;2}^{-2} \mathrm{E} \sum W_{k_1 i_1 j_1}(s) W_{k_1 i_1 l_1}(t) W_{k'_1 i'_1 j'_1}(T_{k_1 i_1 l_1}) W_{k_2 i_2 j_2}(s) W_{k_2 i_2 l_2}(t) W_{k'_2 i'_2 j'_2}(T_{k_2 i_2 l_2})$$

$$(Y \quad)_{k_1 i_1 j_1}(Y \quad)_{k'_1 i'_1 j'_1}(Y \quad)_{k_2 i_2 j_2}(Y \quad)_{k'_2 i'_2 j'_2};$$

$$(b) = \frac{1}{2} S_{K;1}^{-2} S_{K;2}^{-2} \mathrm{E} \sum W_{k_1 i_1 j_1}(s) W_{k_1 i_1 l_1}(t) W_{k'_1 i'_1 j'_1}(T_{k_1 i_1 l_1}) W_{k_2 i_2 j_2}(s) W_{k_2 i_2 l_2}(t) W_{k'_2 i'_2 j'_2}(T_{k_2 i_2 l_2})$$

$$(Y \quad)_{k_1 i_1 j_1}(Y \quad)_{k'_1 i'_1 j'_1}(Y \quad)_{k_2 i_2 j_2} \quad {}''_{k_2 i_2 l_2}(T_{k'_2 i'_2 j'_2} \quad T_{k_2 i_2 l_2})^2;$$

$$(c) = \frac{1}{2} S_{K;1}^{-2} S_{K;2}^{-2} \mathrm{E} \sum W_{k_1 i_1 j_1}(s) W_{k_1 i_1 l_1}(t) W_{k'_1 i'_1 j'_1}(T_{k_1 i_1 l_1}) W_{k_2 i_2 j_2}(s) W_{k_2 i_2 l_2}(t) W_{k'_2 i'_2 j'_2}(T_{k_2 i_2 l_2})$$

$$(Y \quad)_{k_1 i_1 j_1}(Y \quad)_{k_2 i_2 j_2}(Y \quad)_{k'_2 i'_2 j'_2} \quad {}''_{k_1 i_1 l_1}(T_{k'_1 i'_1 j'_1} \quad T_{k_1 i_1 l_1})^2;$$

$$(d) = \frac{1}{4} S_{K;1}^{-2} S_{K;2}^{-2} \mathrm{E} \sum W_{k_1 i_1 j_1}(s) W_{k_1 i_1 l_1}(t) W_{k'_1 i'_1 j'_1}(T_{k_1 i_1 l_1}) W_{k_2 i_2 j_2}(s) W_{k_2 i_2 l_2}(t) W_{k'_2 i'_2 j'_2}(T_{k_2 i_2 l_2})$$

$$(Y \quad)_{k_1 i_1 j_1}(Y \quad)_{k_2 i_2 j_2} \quad {}''_{k_1 i_1 l_1}(T_{k'_1 i'_1 j'_1} \quad T_{k_1 i_1 l_1})^2 \quad {}''_{k_2 i_2 l_2}(T_{k'_2 i'_2 j'_2} \quad T_{k_2 i_2 l_2})^2;$$

which are computed as follows.

a. For (a), there are four cases according to the relationship between $(k_1; i_1)$, $(k_2; i_2)$, $(k'_1; i'_1)$, $(k'_2; i'_2)$ to make the quantity nonzero:

(B.1) Let ${}^{a}_{1} = f(k_1; i_1) = (k'_1; i'_1); (k_2; i_2) = (k'_2; i'_2); (k_1; i_1) \neq (k_2; i_2); j_1 \neq l_1; j_2 \neq l_2 g$,

$$(a{:}1) = S_{K;1}^{-2} S_{K;2}^{-2} \mathrm{E} \sum_{{}^{a}_{1}} W_{k_1 i_1 j_1}(s) W_{k_1 i_1 l_1}(t) W_{k_1 i'_1 j_1}(T_{k_1 i_1 l_1}) W_{k_2 i_2 j_2}(s) W_{k_2 i_2 l_2}(t)$$

$$W_{k_2 i_2 j'_2}(T_{k_2 i_2 l_2})(Y \quad)_{k_1 i_1 j_1}(Y \quad)_{k_1 i'_1 j_1}(Y \quad)_{k_2 i_2 j_2}(Y \quad)_{k_2 i_2 j'_2}$$

$$= S_{K;1}^{-2} S_{K;2}^{-2} \sum_{{}^{a}_{1}} \mathrm{E} \quad W_{k_1 i_1 l_1}(t) W_{k_1 i_1 j_1}(s) W_{k_1 i'_1 j_1}(T_{k_1 i_1 l_1})(Y \quad)_{k_1 i_1 j_1}(Y \quad)_{k_1 i'_1 j_1}$$

$$\mathrm{E}\quad W_{k_2 i_2 l_2}(t) W_{k_2 i_2 j_2}(s) W_{k_2 i_2 j_2'}(T_{k_2 i_2 l_2})(Y \qquad)_{k_2 i_2 j_2}(Y \qquad)_{k_2 i_2 j_2'}$$

$$=(\mathrm{E}\quad \hat{R}_1(s;t))^2.$$

(B.2) Let $\ ^a_2 = \{(k_1;i_1) = (k_2;i_2); (k_1';i_1') = (k_2';i_2'); (k_1;i_1) \neq (k_1';i_1'); j_1 \neq l_1; j_2 \neq l_2; j_1' \neq j_2'\}$,

$$(a{:}2) = S_{K;1}^{-2} S_{K;2}^{-2} \sum_{\ ^a_2} \mathrm{E}\{W_{k_1 i_1 j_1}(s) W_{i_2 j_1 k_1}(s)(Y \qquad)_{k_1 i_1 j_1}(Y \qquad)_{i_2 j_1 k_1}\}$$

$$\left[\mathrm{E}\quad W_{k_1 i_1 l_1}(t) W_{l_2 j_1 k_1}(t) W_{k_1' i_1' j_1'}(T_{k_1 i_1 l_1}) W_{i_2' j_1' k_1'}(T_{l_2 j_1 k_1})(Y \qquad)_{k_1' i_1' j_1'}(Y \qquad)_{i_2' j_1' k_1'} \right]$$

$$= S_{K;1}^{-2} S_{K;2}^{-2} \sum_{\ ^a_2} \quad (s;s) \quad (t;t) f^2(s) f^4(t) + O_p(\quad _{;2} + \quad _{;2})$$

$$= S_{K;1}^{-2} S_{K;2}^{-2} S_{K;3}^{2} \quad (s;s) \quad (t;t) f^2(s) f^4(t) + O_p(\quad _{;2} + \quad _{;2}) \ .$$

(B.3) Let $\ ^a_3 = \{(k_1;i_1) = (k_2';i_2'); (k_2;i_2) = (k_1';i_1'); (k_1;i_1) \neq (k_2;i_2); j_1 \neq l_1; j_2 \neq l_2\}$,

$$(a{:}3) = S_{K;1}^{-2} S_{K;2}^{-2} \mathrm{E} \Big[\sum_{\ ^a_3} W_{k_1 i_1 j_1}(s) W_{k_1 i_1 l_1}(t) W_{k_2 i_2 j_1'}(T_{k_1 i_1 l_1})$$

$$W_{k_2 i_2 j_2}(s) W_{k_2 i_2 l_2}(t) W_{k_1 i_1 j_2'}(T_{k_2 i_2 l_2})$$

$$(Y \qquad)_{k_1 i_1 j_1}(Y \qquad)_{k_2 i_2 j_1'}(Y \qquad)_{k_2 i_2 j_2}(Y \qquad)_{k_1 i_1 j_2'} \Big]$$

$$=(\mathrm{E}\quad \hat{R}_1(s;t))^2.$$

(B.4) Let $\ ^a_4 = \{(k_1;i_1) = (k_2;i_2) = (k_1';i_1') = (k_2';i_2'); j_1 \neq l_1; j_2 \neq l_2\}$ and omit the subscript ki,

$$(a{:}4) = S_{K;1}^{-2} S_{K;2}^{-2} \mathrm{E} \Big\{ \sum_{\ ^a_4} W_{j_1}(s) W_{l_1}(t) W_{j_1'}(T_{l_1}) W_{j_2}(s) W_{l_2}(t) W_{j_2'}(T_{l_2})$$

$$(Y \qquad)_{j_1}(Y \qquad)_{j_1'}(Y \qquad)_{j_2}(Y \qquad)_{j_2'} \Big\} \ .$$

It is the summation of the following sets,

$\ ^a_{4:1} = \{j_1 = j_2; j_1' = j_2'; l_1 = l_2; j_1 \neq l_1; j_2 \neq l_2\};$

$\ ^a_{4:2} = \{j_1 = j_2; j_1' \neq j_2'; j_1 \neq l_1; j_2 \neq l_2\} \cup \{j_1 \neq j_2; j_1' = j_2'; l_1 = l_2; j_1 \neq l_1; j_2 \neq l_2\};$

$\ ^a_{4:3} = \{j_1; j_2; j_1'; j_2'$ are not equal$; j_1 \neq l_1; j_2 \neq l_2\}.$

Correspondingly, we de ne

$$E_{1|4}(s;t) = \mathrm{E}\quad Y_1 \quad (T_1)\ ^2\ Y_2 \quad (T_2)\ ^2 j T_1 = s; T_2 = t\ ;$$

$$E_{2|4}(s;t) = \mathrm{E}\quad Y_1 \quad (T_1)\ ^2\ Y_2 \quad (T_2) \quad Y_3 \quad (T_3)\ j T_1 = s; T_2 = t; T_3 = t\ ;$$

$$E_{3|4}(s;t) = \mathrm{E}\quad Y_1 \quad (T_1)\quad Y_2 \quad (T_2)\quad Y_3 \quad (T_3)\quad Y_4 \quad (T_4)\ j T_1 = s; T_2 = t; T_3 = s; T_4 = t\ :$$

Then by easy calculation,

$$(a{:}4)_{\substack{a\\4:1}} \doteq S_{K;1}^{-2} S_{K;2}^{-2} \left(\sum_{k=1}^{K} s_{k;3} \mathrm{e}_{;k}^{-1} \mathrm{e}_{;k}^{-2} \right) R(W)^3 E_{1|4}(s;t) f(s) f^2(t);$$

$$(a{:}4)_{\substack{a\\4:2}} \doteq S_{K;1}^{-2} S_{K;2}^{-2} \left\{ \left(\sum_{k=1}^{K} s_{k;5} \mathrm{e}_{;k}^{-1} \right) R(W) E_{2|4}(s;t) f(s) f^4(t) + \left(\sum_{k=1}^{K} s_{k;4} \mathrm{e}_{;k}^{-1} \mathrm{e}_{;k}^{-1} \right) R(W)^2 E_{2|4}(t;s) f^2(s) f^2(t) \right\};$$

$$(a{:}4)_{\substack{a\\4:3}} \doteq S_{K;1}^{-2} S_{K;2}^{-2} S_6 E_{3|4}(s;t) f^2(s) f^4(t):$$

b. For (b) and (c), it is required that $(k_1; i_1) = (k_2; i_2) = (k_1'; i_1')$, de ne

$$E_{2|3}(s;s;t) = \mathrm{E}\quad Y_1 \quad (T_1)\quad Y_2 \quad (T_2)\quad Y_3 \quad (T_3)\ j T_1 = s; T_2 = s; T_3 = t\ :$$

By easy calculation,

$$(b) + (c) \doteq S_{K;1}^{-2} S_{K;2}^{-2}\ (W) E_{2|3}(s;s;t) \left(\sum_{k=1}^{K} s_{k;3} \mathrm{e}_{;k}^{2} \right):$$

c. For (d), we need $(k_1; i_1) = (k_2; i_2)$, then

$$(d) \doteq \frac{1}{4} S_{K;2}^{-2}\ ^2(W) f\ ''(t) g^2\ (s;s) f^2(s) f^4(t) \left(\sum_{k=1}^{K} s_{k;4} \mathrm{e}_{;k}^{4} \right):$$

From the above arguments,

$$\mathrm{Var}\quad R_1 = \mathrm{E}\quad R_1^2 \quad (\mathrm{E}\quad R_1)^2$$
$$= O_p\ S_{K;1}^{-2} S_{K;2}^{-2} \left\{ S_{K;3}^2 + \sum_k s_{k;3} \mathrm{e}_{;k}^{-1} \mathrm{e}_{;k}^{-2} + \sum_k s_{k;4} \mathrm{e}_{;k}^{-1} \mathrm{e}_{;k}^{-1} + \sum_k s_{k;5} \mathrm{e}_{;k}^{-1} \right\} + S_{K;2}^{-2} \sum_k s_{k;4} \mathrm{e}_{;k}^{4}\ :$$

Note that

$$\mathrm{Cov}(Q_{0R_1}, \widehat{R}_1) = S_{K,1}^{-2} S_{K,2}^{-1} \left(\sum_{k=1}^{K} \sum_{i=1}^{n_k} m_{ki}^2 (m_{ki} - 1) \right) {}_{;-1} R(W) (s,t) f(s) f^3(t);$$

$$\mathrm{E}(Q_{0R_1}) = f(t) + \frac{1}{2} (W) f''(t) {}_{;2} + o_p({}_{;2});$$

$$\mathrm{Var}(Q_{0R_1}) = \frac{{}_{;-1}}{S_{K,1}} R(W) f(t) + o_p \left(\frac{{}_{;-1}}{S_{K,1}} \right).$$

By the delta method,

$$\mathrm{Var}\left(\frac{\widehat{R}_1}{Q_{0R_1}} \right) = \frac{\mathrm{Var}\, \widehat{R}_1}{\mathrm{E}^2 Q_{0R_1}} - 2\frac{\mathrm{E}\, \widehat{R}_1 \,\mathrm{Cov}(Q_{0R_1}, \widehat{R}_1)}{\mathrm{E}^3 Q_{0R_1}} + \frac{\mathrm{E}^2\, \widehat{R}_1 \,\mathrm{Var}\, Q_{0R_1}}{\mathrm{E}^4 Q_{0R_1}}$$

$$= O_p \Big\{ S_{K,1}^{-2} S_{K,2}^{-2} \Big(S_{K,3}^2 + \sum_k s_{k,3} e_{;k}^{-1} e_{;k}^{-2} + \sum_k s_{k,4} e_{;k}^{-1} e_{;k}^{-1} + \sum_k s_{k,5} e_{;k}^{-1} \Big) + S_{K,2}^{-2} \sum_k s_{k,4} e_{;k}^4 \Big\}.$$

∎

Now we give the proof of the asymptotic distribution of $e^{(K)}$ in Theorem 2.

Proof. Denote

$$f_1 = Q_{20} Q_{02} - Q_{11}^2, \quad f_2 = Q_{10} Q_{02} - Q_{01} Q_{11}, \quad f_3 = Q_{01} Q_{20} - Q_{10} Q_{11}.$$

Let e be the local linear estimate based on $\widehat{C}_{kijl} = (Y_{kij} - e_{kij})(Y_{kil} - e_{kil})$ which can be expressed as

$$e = \frac{f_1 R_{00} - f_2 R_{10} - f_3 R_{01}}{f_1 Q_{00} - f_2 Q_{10} - f_3 Q_{01}},$$

where for $p, q = 0, 1, 2$,

$$Q_{p,q} = S_{K,2}^{-1} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{1 \le j \ne l \le m_{ki}} \tilde{W}_{;k}(T_{kij} - s) \tilde{W}_{;k}(T_{kil} - t) \left(\frac{T_{kij} - s}{e_{;k}} \right)^p \left(\frac{T_{kil} - t}{e_{;k}} \right)^q$$

$$= \sum_k \omega_k Q_{pq;k},$$

$$R_{p,q} = S_{K,2}^{-1} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{1 \le j \ne l \le m_{ki}} \tilde{W}_{;k}(T_{kij} - s) \tilde{W}_{;k}(T_{kil} - t) \left(\frac{T_{kij} - s}{e_{;k}} \right)^p \left(\frac{T_{kil} - t}{e_{;k}} \right)^q \widehat{C}_{kijl}$$

$$= \sum_k \omega_k R_{pq;k}.$$

Let

$$\tilde{e}_{1,k} = \frac{1}{\tilde{e}_{,k}} \frac{f_2(R_{10} - R_{00}Q_{10}/Q_{00})}{f_1 Q_{10} - f_2 Q_{10}^2/Q_{00} - f_3 Q_{01}Q_{10}/Q_{00}},$$

$$\tilde{e}_{2,k} = \frac{1}{\tilde{e}_{,k}} \frac{f_3(R_{01} - R_{00}Q_{01}/Q_{00})}{f_1 Q_{01} - f_2 Q_{01}Q_{10}/Q_{00} - f_3 Q_{01}^2/Q_{00}},$$

$$\tilde{e}^*_{1,k} = \frac{1}{\tilde{e}_{,k}} \frac{f_2(R^*_{10} - R^*_{00}Q_{10}/Q_{00})}{f_1 Q_{10} - f_2 Q_{10}^2/Q_{00} - f_3 Q_{01}Q_{10}/Q_{00}},$$

$$\tilde{e}^*_{2,k} = \frac{1}{\tilde{e}_{,k}} \frac{f_3(R^*_{01} - R^*_{00}Q_{01}/Q_{00})}{f_1 Q_{01} - f_2 Q_{01}Q_{10}/Q_{00} - f_3 Q_{01}^2/Q_{00}},$$

$$R^*_{p,q} = S_{K,2}^{-1} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{1\le j\ne l\le m_{ki}} \tilde{W}_{,k}(T_{kij} - s)\tilde{W}_{,k}(T_{kil} - t) \left(\frac{T_{kij} - s}{\tilde{e}_{,k}}\right)^p \left(\frac{T_{kil} - t}{\tilde{e}_{,k}}\right)^q C_{kijl},$$

$$\hat{C}_{pq} = S_{K,2}^{-1} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{1\le j\ne l\le m_{ki}} \tilde{W}_{,k}(T_{kij} - s)\tilde{W}_{,k}(T_{kil} - t) T_{kij}^p T_{kil}^q = \sum_k \omega_k \hat{C}_{pq,k}.$$

Then we can write

$$\tilde{e} = \frac{R_{00}}{Q_{00}} - \sum_k \omega_k \tilde{e}_{1,k} \frac{\hat{C}_{10,k} - sQ_{00}}{Q_{00}} - \sum_k \omega_k \tilde{e}_{2,k} \frac{\hat{C}_{01,k} - tQ_{00}}{Q_{00}},$$

$$\bar{} = \frac{R_{00}}{Q_{00}} - \frac{\partial}{\partial s} \frac{\hat{C}_{10} - sQ_{00}}{Q_{00}} - \frac{\partial}{\partial t} \frac{\hat{C}_{01} - tQ_{00}}{Q_{00}},$$

$$\bar{}^* = \frac{R^*_{00}}{Q_{00}} - \frac{\partial}{\partial s} \frac{\hat{C}_{10} - sQ_{00}}{Q_{00}} - \frac{\partial}{\partial t} \frac{\hat{C}_{01} - tQ_{00}}{Q_{00}}.$$

Following similar arguments in the proof of Theorem 1, we can prove that

$$\tilde{e} - \bar{} = o_p \left[\min \left\{ \left(S_{K,2}^{-2} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \frac{s_{k,3}}{\tilde{e}_{,k}} \right)^{\frac{1}{2}}, \; S_{K,4}/S_{K,2}^{2}{}^{\frac{1}{2}} \right\} \right].$$

Now we derive the asymptotic distribution of $\bar{}$. We first show that $\bar{} - \bar{}^* = O_p\{(N_K)^{-1}\}$.

(B.1) *Calculate* $\bar{} - \bar{}^*$.

From the definition of $\bar{}$ and $\bar{}^*$, we have $\bar{} - \bar{}^* = R_{00}/Q_{00}$. Note that

$$\begin{aligned} R_{00} =& S_{K,2}^{-1} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j\ne l} W_{kij}(s) W_{kil}(t) \hat{C}_{kijl} \\ =& S_{K,2}^{-1} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j\ne l} W_{kij}(s) W_{kil}(t) \{(Y_{kij} - {}_{kij})(Y_{kil} - {}_{kil}) + (Y_{kij} - {}_{kij})({}_{kil} - e_{kil}) \\ & + ({}_{kij} - e_{kil})(Y_{kil} - {}_{kil}) + (e_{kij} - {}_{kij})(e_{kil} - {}_{kil})\} \end{aligned}$$

$$\doteq R^*_{00} + S^{-1}_{K;2} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j \neq l} W_{kij}(s) W_{kil}(t) f(Y_{kij} \quad _{kij})(\quad _{kil} \quad \text{e}_{kil})$$

$$+ (\quad _{kij} \quad \text{e}_{kil})(Y_{kil} \quad _{kil}) g.$$

Recall that in S.1, we have

$$\text{e}(t) \quad \bar{}(t) = o_p(\bar{}(t) \quad (t));$$

where

$$\bar{}(t) = \frac{R_0}{Q_0} \quad \frac{\tilde{G}_1}{Q_0} \; '(t) + t \frac{\tilde{G}_0}{Q_0} \; '(t);$$

$$Q_r(t) = \frac{1}{S_{K;1}} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j=1}^{m_{ki}} \tilde{W}_{;k}(T_{kij} \quad t) \quad \frac{T_{kij} \quad t}{\text{e}_{;k}} \; ^r;$$

$$R_r(t) = \frac{1}{S_{K;1}} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j=1}^{m_{ki}} \tilde{W}_{;k}(T_{kij} \quad t) \quad \frac{T_{kij} \quad t}{\text{e}_{;k}} \; ^r Y_{kij};$$

$$\tilde{G}_r(t) = \frac{1}{S_{K;1}} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j=1}^{m_{ki}} \tilde{W}_{;k}(T_{kij} \quad t) T^r_{kij}.$$

then one can obtain

$$S^{-1}_{K;2} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j \neq l} W_{kij}(s) W_{kil}(t)(Y_{kij} \quad _{kij})(\quad _{kil} \quad \text{e}_{kil})$$

$$\doteq S^{-1}_{K;2} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j \neq l} W_{kij}(s) W_{kil}(t)(Y_{kij} \quad _{kij})(\quad _{kil} \quad \bar{}_{kil})$$

$$= S^{-1}_{K;2} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j \neq l} W_{kij}(s) W_{kil}(t)(Y_{kij} \quad _{kij}) \quad (T_{kil}) \quad \frac{R_0}{Q_0} + \frac{\check{G}_1}{Q_0} \; '(T_{kil}) \quad T_{kil} \frac{\check{G}_0}{Q_0} \; '(T_{kil})$$

$$= S^{-1}_{K;2} Q_0^{-1} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j \neq l} W_{kij}(s) W_{kil}(t)(Y_{kij} \quad _{kij})$$

$$Q_0(T_{kil}) \; (T_{kil}) \quad R_0(T_{kil}) + \check{G}_1(T_{kil}) \; '(T_{kil}) \quad T_{kil} \check{G}_0(T_{kil}) \; '(T_{kil})$$

$$= S^{-1}_{K;1} S^{-1}_{K;2} Q_0^{-1} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j \neq l} \sum_{k'=1}^{K} \sum_{i'=1}^{n_{k'}} \sum_{j'=1}^{m_{k'i'}} \tilde{W}_{;k'}(T_{kij} \quad s) \tilde{W}_{;k'}(T_{kil} \quad t) \tilde{W}_{;k'}(T_{k'i'j'} \quad T_{kil})$$

$$(Y_{kij} \quad (T_{kij})) f \; (T_{kil}) \quad Y_{k'i'j'} + T_{k'i'j'} \; '(T_{kil}) \quad T_{kil} \; '(T_{kil}) g.$$

Take the Taylor expansion of (T_{kil}) at $(T_{k'i'j'})$ in the last bracket,

$$S^{-1}_{K;2} \sum_{k=1}^{K} \sum_{i=1}^{n_k} \sum_{j \neq l} W_{kij}(s) W_{kil}(t)(Y_{kij} \quad _{kij})(\quad _{kil} \quad \text{e}_{kil})$$

$$\doteq \quad S_{K;1}^{-1}S_{K;2}^{-1}Q_0^{-1}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j\neq l}\sum_{k'=1}^{K}\sum_{i'=1}^{n_{k'}}\sum_{j'=1}^{m_{k'i'}}\tilde{W}_{\;;k'}(T_{kij}\quad s)\tilde{W}_{\;;k'}(T_{kil}\quad t)$$

$$\tilde{W}_{\;;k'}(T_{k'i'j'}\quad T_{kil})(Y_{kij}\qquad (T_{kij}))\Big\{\, Y_{k'i'j'}\qquad (T_{k'i'j'})+$$

$$\;'(T_{kil})\qquad '(T_{k'i'j'})\;(T_{k'i'j'}\quad T_{kil})+\frac{1}{2}\;''(T_{kil})(T_{k'i'j'}\quad T_{kil})^2\Big\}:$$

Then take the Taylor expansion of $\;'(T_{k'i'j'})$ at $\;'(T_{kil})$ in the last bracket,

$$S_{K;2}^{-1}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j\neq l}W_{kij}(s)W_{kil}(t)(Y_{kij}\qquad {}_{kij})(\;{}_{kil}\quad \mathrm{e}_{kil})$$

$$\doteq \quad S_{K;1}^{-1}S_{K;2}^{-1}Q_0^{-1}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j\neq l}\sum_{k'=1}^{K}\sum_{i'=1}^{n_{k'}}\sum_{j'=1}^{m_{k'i'}}\tilde{W}_{\;;k'}(T_{kij}\quad s)\tilde{W}_{\;;k'}(T_{kil}\quad t)$$

$$\tilde{W}_{\;;k'}(T_{k'i'j'}\quad T_{kil})(Y_{kij}\qquad (T_{kij}))\Big\{\, Y_{k'i'j'}\qquad (T_{k'i'j'})\qquad \frac{1}{2}\;''(T_{kil})(T_{k'i'j'}\quad T_{kil})^2\Big\}:$$

Denote

$$\Delta\hat{R}_1 = \quad S_{K;1}^{-1}S_{K;2}^{-1}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j\neq l}\sum_{k'=1}^{K}\sum_{i'=1}^{n_{k'}}\sum_{j'=1}^{m_{k'i'}}\tilde{W}_{\;;k'}(T_{kij}\quad s)\tilde{W}_{\;;k'}(T_{kil}\quad t)$$

$$\tilde{W}_{\;;k'}(T_{k'i'j'}\quad T_{kil})(Y_{kij}\qquad (T_{kij}))\;Y_{k'i'j'}\qquad (T_{k'i'j'})\qquad \frac{1}{2}\;''(T_{kil})(T_{k'i'j'}\quad T_{kil})^2\quad ;$$

$$\Delta\hat{R}_2 = \quad S_{K;1}^{-1}S_{K;2}^{-1}\sum_{k=1}^{K}\sum_{i=1}^{n_k}\sum_{j\neq l}\sum_{k'=1}^{K}\sum_{i'=1}^{n_{k'}}\sum_{j'=1}^{m_{k'i'}}\tilde{W}_{\;;k'}(T_{kij}\quad s)\tilde{W}_{\;;k'}(T_{kil}\quad t)$$

$$\tilde{W}_{\;;k'}(T_{k'i'j'}\quad T_{kij})(Y_{kil}\qquad (T_{kil}))\;Y_{k'i'j'}\qquad (T_{k'i'j'})\qquad \frac{1}{2}\;''(T_{kij})(T_{k'i'j'}\quad T_{kij})^2\quad ;$$

$$Q_{0R1} = Q_0(T_{kil});\quad Q_{0R2} = Q_0(T_{kij}):$$

It can be written as

$$R_{00} \doteq R_{00}^{*} + \frac{\hat{R}_1}{Q_{0R_1}} + \frac{\hat{R}_2}{Q_{0R_2}};$$

The last terms in the right hand side are of the same order. By Lemma S.1, we have

$$\mathrm{E}(\;\hat{R}_1 = Q_{0R_1}) = O_p(S_{K;1}^{-1}S_{K;2}^{-1}S_{K;3});$$

$$\mathrm{Var}(\;\hat{R}_1 = Q_{0R_1}) = O_p\;\; S_{K;1}^{-2}S_{K;2}^{-2}\;\; S_{K;3}^{2}(1 + S_{K;1}^{-1}\quad {}_{;-1}) + S_{K;3}\quad {}_{;-1}\quad {}_{;-2}$$

$$+ S_4\quad {}_{;-1}\quad {}_{;-1} + S_5\quad {}_{;-1} + S_6\quad + S_{K;2}^{-2}S_4\,(\quad {}_{;2})^2\quad :$$

Hence

$$-\quad -^{*} = O_p\;\; S_{K;1}^{-1}S_{K;2}^{-1}\;\; S_{K;3}^{2}(1 + S_{K;1}^{-1}\quad {}_{;-1}) + S_{K;3}\quad {}_{;-1}\quad {}_{;-2}$$

$$+ S_{K;4}\quad {}_{;-1}\quad {}_{;-1} + S_5\quad {}_{;-1} + S_6\quad {}^{\frac{1}{2}} + S_{K;2}^{-1}S_4^{\frac{1}{2}}\,(\quad {}_{;2})\quad :$$

(B.2) *Asymptotic distribution of* $\bar{\ }^{*}$.

De ne

$$\bar{\ }^{*} = \frac{R^{*}_{00}}{Q_{00}} \quad \frac{@}{@s} \quad \frac{\tilde{G}_{10} \quad sQ_{00}}{Q_{00}} \quad \frac{@}{@t} \quad \frac{\tilde{G}_{01} \quad tQ_{00}}{Q_{00}}:$$

The asymptotic normality of $\bar{\ }^{*}$ is accomplished by the asymptotic joint normality of $(R^{*}_{00} \quad ER^{*}_{00}; \tilde{G}_{00} \quad E\tilde{G}_{00}; \tilde{G}_{10} \quad E\tilde{G}_{10}; \tilde{G}_{01} \quad E\tilde{G}_{01})$. Speci cally, let

$$E_1(s;t) = E f(Y_1 \quad (T_1))^2 (Y_2 \quad (T_2))^2 j T_1 = s; T_2 = tg;$$

$$E_2(s;t) = E f(Y_1 \quad (T_1))^2 (Y_2 \quad (T_2))(Y_2 \quad (T_3)) j T_1 = s; T_2 = t; T_3 = tg:$$

By easy calculation, we have

$$E\tilde{G}_{pq} = s^p t^q f(s) f(t) + \frac{1}{2} \quad (W) f s^p t^q f''(s) f(t) + s^p t^q f(s) f''(t) g \quad \bar{\ }^{2}$$

$$+ \frac{1}{2} \quad (W) f 2p f'(s) f(t) + 2q f'(t) f(s) g \quad \bar{\ }^{2} + o_p \quad \bar{\ }^{2} \quad ;$$

$$ER^{*}_{00} = \quad (s;t) f(s) f(t) + \frac{1}{2} \quad (W) \bar{\ }^{2} \quad \frac{@^2}{@s^2}(s;t) f(s) f(t)$$

$$+ 2\frac{@}{@s}(s;t) f'(s) f(t) + \quad (s;t) f''(s) f(t) + \frac{@^2}{@t^2}(s;t) f(t) f(s)$$

$$+ 2\frac{@}{@}(s;t) f'(t) f(s) + \quad (s;t) f''(t) f(s) \quad + o_p \quad \bar{\ }^{2} \quad ;$$

$$\mathrm{Var}(R^{*}_{00}) = f1 + I(s=t) g \quad R(W)^2 E_1(s;t) f(s) f(t) S^{-2}_{K;2} \bar{\ }^{-2}$$

$$+ R(W) \quad f^2(s) f(t) E_2(t;s) + f(s) f(t)^2 E_2(s;t) \quad S^{-1}_{K;2} \sum_{k=1} \frac{s_{k;3}}{e_{;k}}$$

$$+ V_3(s;t) f^2(s) f^2(t) S^{-2}_{K;2} S_4 + o \quad S^{-2}_{K;2} \bar{\ }^{-2}$$

$$+ o_p \quad S^{-2}_{K;2} \sum_{k=1} \frac{s_{k;3}}{e_{;k}} \quad + o_p \quad S^{-2}_{K;2} S_4 \quad ;$$

$$\mathrm{Cov}(\tilde{G}_{pq}; \tilde{G}_{p'q'}) = f1 + I(s=t) g s^{p+p'} t^{q+q'} \quad R(W)^2 f(s) f(t) S^{-2}_{K;2} \bar{\ }^{-2}$$

$$+ R(W) f f^2(s) f(t) + f(s) f^2(t) g S^{-2}_{K;2} \sum_{k=1} \frac{s_{k;3}}{e_{;k}}$$

$$+ o_p \quad S^{-2}_{K;2} \bar{\ }^{-2} \quad + o_p \quad S^{-2}_{K;2} \sum_{k=1} \frac{s_{k;3}}{e_{;k}} \quad ;$$

$$\mathrm{Cov}(\tilde{G}_{pq}; R^{*}_{00}) = f1 + I(s=t) g s^p t^q \quad R(W)^2 \quad (s;t) f(s) f(t) S^{-2}_{K;2} \bar{\ }^{-2}$$

$$+ R(W)\ (s;t)\{f^2(s)f(t) + f(s)f^2(t)\}S_{K;2}^{-2}\sum_{k=1}^{K}\frac{s_{k;3}}{e_{;k}}\Bigg]$$

$$+ o_p\left(S_{K;2}^{-2}\bar{\ }^{-2}\right) + o_p\left(S_{K;2}^{-2}\sum_{k=1}^{K}\frac{s_{k;3}}{e_{;k}}\right).$$

From the delta method, $\bar{\ }(t)$ follows the asymptotic normality that

$$\bar{\ }_*^{-1/2}\left[\bar{\ }^*(s;t) - (s;t) - \frac{1}{2}\bar{\ }^2(W)\left(\frac{\partial^2}{\partial s^2}(s;t) + \frac{\partial^2}{\partial t^2}(s;t)\right) + o_p\left(\bar{\ }^2\right)\right] \xrightarrow{d} N(0,1),$$

where

$$\bar{\ }_* = \{1 + I(s=t)\}\Bigg[S_{K;2}^{-1}\bar{\ }^{-2}R(W)^2\frac{V_1(s;t)}{f(s)f(t)}$$

$$+ S_{K;2}^{-2}\sum_{k=1}^{K}\frac{s_{k;3}}{e_{;k}}R(W)\frac{f(s)V_2(t;s) + f(t)V_2(s;t)}{f(s)f(t)}\Bigg) + V_3(s;t)S_4/S_{K;2}^2.$$

The proof is completed combining the above arguments. ■

S.3. PROOF OF THEOREM 3

We mention that Theorem 3 and Theorem 4 can be extended to the standard d-dimensional local linear regression. To see this, we introduce the following notations and assumptions.

Notations and assumptions for d-dimensional local linear regression. The underlying model is $Y = m(X) + \varepsilon$ where $X \in R^d$, $E\varepsilon = 0$ and $\mathrm{Var}\,\varepsilon = \ ^2(X)$. Suppose we observe $\{(Y_{ki}, X_{ki} : i = 1, \dots, n_k)\}$ in the kth block and $N_K = \sum_{k=1}^{K} n_k$, i.e., we use n_K to denote the sub-sample size of the Kth data block and $N_K = \sum_{k=1}^{K} n_k$ to denote the full sample size up to time K, which correspond to $s_{K;1}, S_{K;1}$ in (6) for mean estimation and $s_{K;2}, S_{K;2}$ for covariance estimation, respectively. We impose the following assumptions which are in parallel to (A.1)–(A.3) and (A.6).

(B.1) Observations $\{X_{ki} : i = 1, \dots, n_k; k = 1, \dots, K\}$ are i.i.d. copies of a random variable X de ned on $[0,1]^d$ whose density $f(\ )$ is bounded away from 0 with bounded second derivative.

(B.2) The second and fourth derivatives of regression function m(x) are bounded and continuous.

(B.3) Noises ε_{Ki} are i.i.d. with mean 0 and variance $\sigma^2(x)$.

(B.4) The block size $Kn_k/N_K \to 1$ as $K \to \infty$ for $k = 1, 2, \dots, K$.

The optimal bandwidth at time K is $h_*^{(K)} \asymp N_K^{-1/(4+d)}$. The candidate sequence is $\{\lambda_l^{(K)}\}_{l=1}^{L}$ and the pseudo-bandwidths are $\{\widetilde{\eta}_k^{(K)} : k = 1, \dots, K\}$ for $j = 1, \dots, d$. The centroids for the candidate sequence are $\{\eta_i^{(K)} : i = 1, \dots, L\}$. Further define

$$\eta_i^{(K)} = \sum_{k=1}^{K} \frac{n_k}{N_K} \left(\widetilde{\eta}_k^{(K)}\right)^{i}.$$

We also mention that the mean and covariance estimates corresponds the cases $d = 1$ and $d = 2$, respectively.

When estimating θ, all observations are pooled together and the index of subjects become invalid, which is intrinsically one-dimensional local linear regression. Hence one can derive the following result for standard d-dimensional local linear regression by the same arguments in Theorem 1.

Lemma S. 2. *Suppose assumptions* (B.1)–(B.3) *and (A.8)–(A.9) hold. If*

$$\widehat{h}_j^{(K)} - h_*^{(K)} = o_p\left(N_K^{-\frac{1}{4+d}}\right), \quad j = 1, \dots, d, \tag{S.3}$$

then a fixed interior point $x \in (0,1)^d$, as $K \to \infty$, the online estimate $\widehat{m}^{(K)}(x)$ satisfies

$$\left\{N_K \eta_{-d}^{(K)}\right\}^{\frac{1}{2}} \left[\widehat{m}^{(K)}(x) - m(x) - \frac{1}{2}\mu_2(W)\left\{\sum_{j=1}^{d} m_j''(x)\right\}\eta^2 + o_p(\eta^2)\right] \xrightarrow{d} N\left(0, \frac{R(W)^d \sigma^2(x)}{f(x)}\right).$$

We prove in Section S.3 that (S.3) is satisfied by the proposed online method.

Denote $\mathcal{X} = \{X_{ki} \in R^d : k = 1, \dots, K; i = 1, \dots, n_k\}$, $\theta = \int \{\sum_{j=1}^{d} \partial^2 m(x)/\partial x_j^2\} f(x)dx$ and $\tau = \int_{[0,1]^d} R(W)^d \sigma^2(x)dx$. Similarly, we adopt another online local cubic and online local linear to estimate θ and τ, which are denoted as $\widetilde{\theta}^{(K)}$ and $\widetilde{\tau}^{(K)}$, respectively. Then the optimal bandwidth and the online estimated bandwidth for $m(\cdot)$ are

$$h_*^{(K)} = \left(\frac{\tau}{\mu^2(W)\theta}\right)^{\frac{1}{4+d}} N_K^{-\frac{1}{4+d}}, \quad \widehat{h}^{(K)} = \left(\frac{\widetilde{\tau}^{(K)}}{\mu^2(W)\widetilde{\theta}^{(K)}}\right)^{\frac{1}{4+d}} N_K^{-\frac{1}{4+d}}. \tag{S.4}$$

The candidate sequence for θ and τ are $\{\lambda_{\theta,l}^{(K)}\}_{l=1}^{J}$ and $\{\lambda_{\tau,l}^{(K)}\}_{l=1}^{J}$, respectively, and the pseudo-bandwidths are $\{\widetilde{\eta}_{\theta,k}^{(K)} : k = 1, \dots, K\}$ and $\{\widetilde{\eta}_{\tau,k}^{(K)} : k = 1, \dots, K\}$. Then we have the following conclusion.

Lemma S. 3. *Suppose assumptions* (B.1){(B.4) *and (A.8){(A.9) hold. When the online bandwidths and candidate sequence for and are*

$$h^{(K)} = GN_K^{-1=(6+d)}; \quad h^{(K)} = RN_K^{-1=(6+d)}; \quad 1 < G; R < 1;$$

$$ {}^{(K)}_{;J} < \quad < {}^{(K)}_{;1} = h^{(K)}; \quad {}^{(K)}_{;J} < \quad < {}^{(K)}_{;1} = h^{(K)};$$

then as $K \; ! \; 1$, $\hat{h}^{(K)}$ *in* (S.4) *satis es*

$$\hat{h}^{(K)} \quad h_*^{(K)}$$

and then $\mathrm{e}^{(K)} \quad = O_p(S_{K;1}^{-2=5})$. For , one can prove that the convergence rate of $\mathrm{e}^{(K)}$ is dominated by $\mathrm{E}_K \quad (s) \quad (t)$ in step 3 of case (b) in Appendix B which equals $O_p(S_{K;2}^{-1=3})$. For dense data, we need only check the order of e_K^2. From Zhang and Chen (2007), $''_{kij} \quad ''_{kij}$ in step 2 of (c) in Appendix B is of order $(S_{K;1})^{-2=25}$, recall that b_{kij}^2 is a debiased version of $''_{kij}$ and e_K^2 is the average of b_{kij}^2, then $\mathrm{e}_K^2 \quad {}^2 = O_p \; S_{K;1}^{-12=25}$. In all the cases, estimates of are dominates those of , which completes the proof of Theorem 3.

S.4. PROOF OF THEOREM 4

Note that the asymptotic distributions of e and e have the consistent form with the standard d-dimensional local linear regression, i.e., the bias is of $O_p(h^2)$ and the variance is of $O_p(N^{-1}h^{-d})$, where $d = 1$ for mean estimation and $d = 2$ for covariance estimation. The lower bound of the relative e ciency also holds for the standard d-dimensional local linear regression. We adopt the notations of the d-dimensional local linear in the proof for notation conciseness.

Proof. Recall that $\mathrm{\hbar}^{(K)} = {}_1^{(K)} > ::: > {}_L^{(K)}$, without loss of generality, we assume ${}_l^{(K)} = g(l)\mathrm{\hbar}^{(K)}$ where $g(l) \quad g(1) = 1$, $l = 1; ::: ; L$. Note that $\mathrm{\hbar}^{(K)} \quad \mathrm{\hbar}^{(K)} = o_p \; N_K^{-1=(d+4)}$ and the candidate bandwidth sequence satis es

$$\frac{\mathrm{e}_k^{(K)}}{\mathrm{\hbar}^{(K)}} = g(l) \quad \frac{N_k}{N_K}^{-\frac{1}{d+2}} + o_p(1):$$

Write $g(l) = fg(l)^{1=} \; g$ and note that $N_k{=}N_K$ grows linearly with respect to k, then shall be $1{=}(d + 2)$ and the optimal $g(l)^{1=}$ shall be linear between $(0; 1)$. Hence, the optimal ${}_l$ is as in (19).

Next we derive the asymptotic lower bound for the relative e ciency of our local linear estimator. Writing $\mathrm{\hbar}^{(K)} = h_*^{(K)} f1 + O_p(N_K^{-2=(d+6)})g$, one can derive

$$eff(\mathrm{m})^{-1} = \frac{d}{d+4} \begin{Bmatrix} \displaystyle\sum_{k=1}^{K} \frac{n_k}{N_K} \left(\frac{\mathrm{e}_k^{(K)}}{\mathrm{\hbar}^{(K)}} \right)^{2} \end{Bmatrix}^{2} + \frac{4}{d+4} \begin{Bmatrix} \displaystyle\sum_{k=1}^{K} \frac{n_k}{N_K} \left(\frac{\mathrm{e}_k^{(K)}}{\mathrm{\hbar}^{(K)}} \right)^{-1} \end{Bmatrix} + O_p \; N_K^{-\frac{2}{6+d}} : \tag{S.5}$$

As shown in Figure 1, when K tends large, there exists a breakpoint K_0: when $k \quad K_0$, e_k equals the last candidate $(1{=}L)^{1=(d+4)}\mathrm{\hbar}_k$ for the limit number of candidates and for $k > K_0$, there are sufficient and close candidates to make a choice. Hence

$$\frac{\tilde e_k^{(K)}}{\hat h^{(K)}} = \begin{cases} L^{-\frac{1}{d+4}} \left(\frac{N_k}{N_K}\right)^{-\frac{1}{d+4}} + O_p\left(N_k^{-\frac{2}{6+d}}\right); & k \le K_0 \\ \left\{1 + \left(\frac{N_k}{N_K} - \frac{l_k}{L}\right)\right\}^{\frac{1}{d+4}} + O_p\left(N_k^{-\frac{2}{6+d}}\right); & k > K_0, \end{cases}$$

where $l_k = \mathrm{argmin}_{1\le l\le L} |N_k/N_K - l/L|$. Then when $k \le K_0$,

$$\sum_{k=1}^{K_0} \frac{n_k}{N_K}\left(\frac{\tilde e_k^{(K)}}{\hat h^{(K)}}\right)^2 = \frac{d+4}{d+2} L^{-\frac{2}{d+4}} \left(\frac{N_{K_0}}{N_K}\right)^{1-\frac{2}{d+4}} + O_p\left(N_k^{-\frac{2}{6+d}}\right);$$

$$\sum_{k=1}^{K_0} \frac{n_k}{N_K}\left(\frac{\tilde e_k^{(K)}}{\hat h^{(K)}}\right)^{-d} = \frac{d+4}{2d+4} L^{\frac{d}{d+4}} \left(\frac{N_{K_0}}{N_K}\right)^{1+\frac{d}{d+4}} + O_p\left(N_k^{-\frac{2}{6+d}}\right).$$

and when $k > K_0$,

$$\sum_{k=K_0+1}^{K} \frac{n_k}{N_K}\left(\frac{\tilde e_k^{(K)}}{\hat h^{(K)}}\right)^2 = 1 - \frac{N_{K_0}}{N_K} + O_p\left(N_K^{-\frac{2}{6+d}}\right);$$

$$\sum_{k=K_0+1}^{K} \frac{n_k}{N_K}\left(\frac{\tilde e_k^{(K)}}{\hat h^{(K)}}\right)^{-1} = 1 - \frac{N_{K_0}}{N_K} + O_p\left(N_K^{-\frac{2}{6+d}}\right).$$

The property of breakpoint K_0 also guarantees that

$$\left(\frac{1}{L}\right)^{\frac{1}{d+4}} \sum_{k=1}^{K_0} \frac{n_k}{N_{K_0}} \hat h^{(k)} \asymp \hat h^{(K)},$$

which is equivalent to

$$\left(\frac{1}{L}\right)^{\frac{1}{d+4}} \sum_{k=1}^{K_0} \frac{n_k}{N_{K_0}} \frac{\hat h^{(k)}/h_*^{(k)}}{\hat h^{(K)}/h_*^{(K)}} \frac{h_*^{(k)}}{h_*^{(K)}} = \left(\frac{1}{L}\right)^{\frac{1}{d+4}} \sum_{k=1}^{K_0} \frac{n_k}{N_{K_0}} \{1 + O_p(N_K^{-\frac{2}{6+d}})\} \left(\frac{N_k}{N_K}\right)^{\frac{1}{d+4}} \asymp 1,$$

and under Assumption (A.5), we obtain

$$\frac{N_{K_0}}{N_K} \asymp \left(\frac{d+4}{d+3}\right)^{d+4} \frac{1}{L} + O_p\left(N_K^{-\frac{2}{6+d}}\right).$$

Denote $\rho = N_{K_0}/N_K$, one can derive from (S.5) that

$$\begin{aligned} eff(\hat m)^{-1} = &\left[\frac{d}{d+4}\left\{\frac{d+4}{d+2} L^{-\frac{2}{d+4}} \rho^{1-\frac{2}{d+4}} + 1 - \rho\right\}^2\right. \\ &\left. + \frac{4}{d+4}\left\{\frac{d+4}{2d+4} L^{\frac{d}{d+4}} \rho^{1+\frac{d}{d+4}} + 1 - \rho\right\}\right] + O_p\left(N_K^{-\frac{2}{6+d}}\right). \end{aligned} \quad \text{(S.6)}$$

Note that (S.6) is strictly increasing with respect to $\in \left[0, \{(d+4)/(d+3)\}^{d+4}/L\right]$. Hence we have

$$eff(m)^{-1} \quad 1 + \frac{2c_1 d + 4c_2}{d+4} L^{-1} + \frac{d}{d+4} c_1^2 L^{-2},$$

where

$$c_1 = \frac{(d+4)^{d+3}}{(d+3)^{d+2}(d+2)} \left(\frac{d+4}{d+3}\right)^{d+4}, \quad c_2 = \frac{(d+4)^{2d+5}}{(d+3)^{2d+4}(2d+4)} \left(\frac{d+4}{d+3}\right)^{d+4}.$$

This completes the proof of Theorem 4.

■

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