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Dependence Calibration in Conditional Copulas: A Nonparametric Approach

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. T e d de e de ce be ee a d . a ab e a a a a c . I a ca e , e e de e de ce be ee e a d . a ab e . a e acc d e . a e a ea ed c . a a e. We e e ce e a a a c d a c a de e e e c a c be a a a e c c a a a d e c a a a e e . a e e c . a a e. I de e a e e c a e a be ee e c a a a e e a d e c . a a e , e e a a a e c a ac ba ed ca e d. O a ce a e c ce e c a a a be e e e a e e da a. T e ed a e a a ead a e c a e ec e d ba ed c - a da ed ed c e . We de . e e a c b a a d . a a ce e e ca a e a , a d e c c e c de ce e a . T e e- a e e a ce e d e a ed a d e a d a ed a b e e Ma c ed M e B da a.

E DS: C a a a e e ; C a e ec ; C . a a e ad e ; L ca e d; L ca a ; P ed c e .

1. Introduction

U de a d de e de ce a a , e c a e , a . a a e a ca de . O e e eed ec a c e x d b a d a ab e a e a c e e e e de e de ce c e. T e c a- e e c c c a a e d b ca be ca ed ced e e ac a de e a a e e a a c e a d b de e- de ce c e. S a ' e e (1959) ce a e e- e ca da eeded e e c a a a e a a . a a e d b ca be c a ac e ed b a a d b a d a c a, a , a . a a e d b c a [0, 1] a a .

I a , e c e b a a e ca e c , ea e be e x e dab e e a d e . Le Y_1 a d Y_2 be c a d . a ab e e e d b c H a d a a d- b F_1 a d F_2, e ec e . S a ' e e e e e e x e ce a e c a $C : [0,1]^2 \to [0,1]$, c a e $H(y_1, y_2) = C\{F_1(y_1), F_2(y_2)\}$, a $(y_1, y_2) \in \mathbb{R}^2$.

I e a 20 ea , c a a e bee de ed a a- e a ed . We e e e eade E b ec , McNe , a d S a a (2002); C e b , L ca , a d Vecc a (2004); a d F ee a d Va de (1998) a ca ec c a ce a d a ce. I . a a a , C a (1978), S a d L (1995), Wa a d We (2000), a d e - a b H a a d (2000) e e e c a ec e de a a e e- -e e da a a d c e . A a d ec e e de a cab , a a e - be a a e c a e c a , ca de x ed b a ea- a ed a a e e θ, a e bee ed e e a e e e e d e e de e de ce a e . W e e c a a de c be e c a , eac a

e c a a a e e , θ, a c e e e de e de ce. A c e e e . e d c c a a d e e e ca be d Ne e (2006) a d e c - ec be ee a c a a d de e de ce c ce a e d c ed de a b J e (1997). I a a a e c a ed e c a c , e a ca be ac e ed a x e d e a e e c a a a e e (Ge e , G d , a d R e , 1995; J e, 1997). A - e a e , e a ca be e ed a a e- ca b e e e a (Fe a a a d Sca e , 2003; C e a d H a , 2007).

A c a a e bee e e a ed a- ca e a e e a 20 ea , e c . a a e ad- e c a a bee c de ed ece . T e e x e S a ' e e c d a d b (Pa , 2006) a ad c . a a e . F a ce, , add Y_1 a d Y_2, e a e a ac . a- a e X, e e e ce X e de e de ce be ee Y_1 a d Y_2 ca be de ed b e c d a c a $C(\cdot \mid X)$, c e d b c $U_1 \equiv F_{1\mid X}(Y_1 \mid x)$ a d $U_2 \equiv F_{2\mid X}(Y_2 \mid x)$ e $X = x$, e e $Y_k \mid X = x$ a cd $F_{k\mid X}(. \mid x), k = 1, 2$. Pa (2006) ed a eac x e X, e c d a d b e de ed b

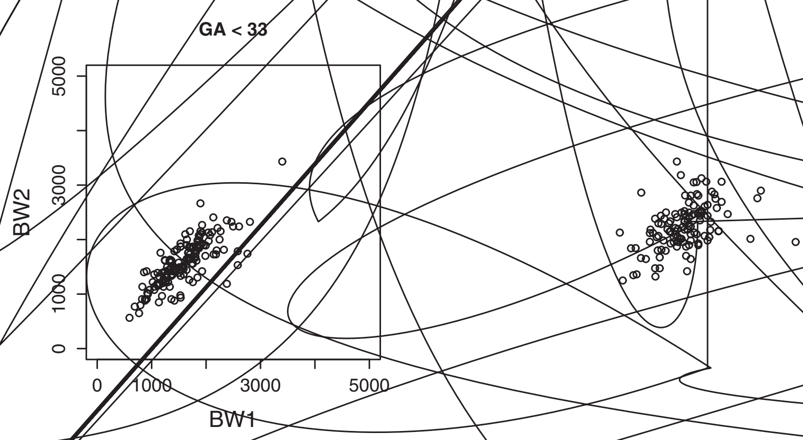
$$H_X(y_1, y_2 \mid x) = C\{F_{1\mid X}(y_1 \mid x), F_{2\mid X}(y_2 \mid x) \mid x\}, \quad \text{a } (y_1, y_2) \in \mathbb{R}^2. \tag{1}$$

C d a c a a e bee ed e c e x a ca e e e a e- a a e de- e de ce c e . a e d e e ce ARMA de (J dea a d R c e , 2006; Pa , 2006; Ba a , Ta - , a d Wa , 2007).

T e a c b e c e de a a a e c ced e e a e e c a ea- be ee e c a a a e e a d e c a a e( ). O a ea e a a e c a ab ea . S ce a a a e c de d ea- e e da a a a e a ead c a ed a e de, a a e c a ac e be ade a e e de e de ce c e d e a a ec ce ed c a c .

I Sec 4, e e a e e ac e a a a e e de e de ce be ee b e a b- e e Ma c ed M e B Da a Se . A e e b , e c de e e e de e ed be ee 28 a d 42 ee e a a d c b ed e ea e. O a e a , F e 1, d ca e a ea e e de e de ce be ee e b e (a) e e e (28 32 ee) a d e (38 42 ee) c a ed e de- e ed a e (33 37 ee). T e e eed c a a e c a ac a a e a de ed e de c a e a be ee e c a a- a e e a d e c a a e.

S e d c e a a e bee b- a a d ed a be . I a ce, e e e ca a a e (ee Fa a d G be , 1996, a c e e e e e) e c a a e ad ed c a e a a ca e d-ba ed de (T b a a d Ha e, 1987). I ac ce, a e e a e d c a be acc a ed b a a e e ec a a - be c a a e e e a be a a e e da a a a d. C a a a e a c a a e e da a c a e a d a ece a ac ed c de ab e ee . S e e d c a e ec - c de d e - e ba ed e e ca c a (D e a , N e ba , a d R ca , 2000), e Ke da ce (Ge e a d R e , 1993; Wa a d We , 2000; Ge - e , Re a d, a d Bea d , 2007), a d e e de e- a (Fe a a , 2005; C a a d C a , 2008). O e - a ced e a a ead a e c a e ec e d ba ed c - a da ed ed c e (CVPE). Be de be da a-ada e, e ed e ec c e a e c a ac c a a e be d e e e a a cab .



T e a ce a ed a . I Sec 2, e e e e ed e a ced e, d c a ec e a ed c a e ec , a d de e e a c b a a d a a ce e a a e c e a ed c c - e c de ce ba d . Sec 3 c a a d- e a d Sec 4 e e e Ma c ed M e B Da a Se e a e e e ce e e a a a e e e de e de ce be ee e b e . D - c a d c c a e e e ed Sec 5. Tec ca de a a d add a a ed a d ea da a e a e ca be d e Web A e d .

2. Methodology

2.1 *Proposed Method*

Le Y_1 a d Y_2 be c a ab e e e a d X be a c a ab e a a a ec e de e de ce be ee Y_1 a d Y_2. We c de e de (1) e c d a de $h_X(Y_1, Y_2 \mid x; \theta, \alpha_1, \alpha_2)$ c a e e e e c d a c a a a e e θ, e e c d a a a de e $f_{1|X}$ a d $f_{2|X}$ a e c a ac e ed b α_1 a d α_2, e ec e ,

$$h_X(y_1, y_2 \mid x; \theta, \alpha_1, \alpha_2) = f_{1|X}(y_1 \mid x; \alpha_1)\, f_{2|X}(y_2 \mid x; \alpha_2) \times c(u_1, u_2 \mid x; \theta, \alpha_1, \alpha_2),$$

e e $u_k = F_{k|X}(y_k \mid x; \alpha_k), k = 1, 2$ a d $c(u_1, u_2 \mid x; \theta, \alpha_1, \alpha_2)$ e c d a c a de . He e, e e a a e e e a e a a e e a e e a a a e d e e ec a a a e e. I ea ee a c a e e e e c e, a ce, a e- e e , a a a c e d ea e ec a d e c a c a a ce c e. He ce, e e a- ca be e ed a e, e a a a a e e a d e e e a. T e, b e ac e e a e $\hat{F}_{1|X}(y_1 \mid x)$ a d $\hat{F}_{2|X}(y_2 \mid x)$ (1), e ca e a e e c a e c a a a e e.

S ce a c e de e de ce c e, e a e a e c d a a a d b $F_{1|X}$ a d $F_{2|X}$ a e , a d c de e de:

$$(U_{1i}, U_{2i}) \mid X_i \sim C\{u_{1i}, u_{2i} \mid \theta(x_i)\},$$

e e $\theta(x_i) = g^{-1}\{\eta(x_i)\}, \quad i = 1, \ldots, n.$

He e, $g^{-1} : \mathbb{R} \rightarrow$ e . e e c , c e e a ec a a a e e a e c ec a e, a d η e *calibration function* be e a ed. T e e *calibration* e a e a e e e de e de ce ad ed e c a a e e ec e c a a a e e . A a e e e a ed ea de e eed c e a a - a e c , ce e e a a ee a e e a e θ e c ec a a e e a e e a c- a c a a de c de a . F a ce, e e e e be e da a a a d. F e q c a ﬂa , e ba d d e ec ce e d e a ba d d h_q^*. F eac e ﬂ- a e (U_{1i}, U_{2i}, X_i), e ba e e a e ﬂ e c d a c a' a a e e $\hat{\theta}_{h_q^*}^{(-i)}$, c , , ead e *best candidate model* ﬂ e q ﬂa , $C_q\{U_{1i}, U_{2i} \,|\, \hat{\theta}_{h_q^*}^{(-i)}(X_i)\}$, $i = 1, \ldots, n, q = 1, \ldots, Q$. We e e c d a e ec a ﬂ a ea e e ed c e ab ﬂ eac ﬂ e ca d da e de . W ﬂa $\mathcal{C}_q$, e *best conditional prediction* ﬂ U_{1i}

$$\widehat{E}_q^{(-i)}(U_{1i} \,|\, U_{2i}, X_i) = \int_0^1 U_1 c_q\left\{U_1, U_{2i} \,|\, \hat{\theta}_{h^*}^{(-i)}(X_i)\right\} dU_1.$$

T e , e CVPE ed de e e de e ec c e

$$\mathrm{CVPE}(\mathcal{C}_q) = \sum_{i=1}^n \left[\left\{U_{1i} - \widehat{E}_q^{(-i)}(U_{1i} \,|\, U_{2i}, X_i)\right\}^2 + \left\{U_{2i} - \widehat{E}_q^{(-i)}(U_{2i} \,|\, U_{1i}, X_i)\right\}^2\right]. \quad (4)$$

T e c a ﬂa $\mathcal{C}_q$, c e d e $\mathrm{CVPE}(\mathcal{C}_q)$ a e, e ec ed. T c e ca be ed a ﬂ . Iﬂ e de e M_0 e e c a ﬂa a d M e c a ﬂa , e e a (4) a ed b $1/n$ a a a ﬂ $E_{M_0}[\{U_1 - E_M(U_1 \,|\, U_2, X)\}^2 \,|\, U_2, X]$, c ed e e de M c ec ec ed, a , $M = M_0$. A a e d ﬂ e ec d a (4).

2.3 *Asymptotic Properties*

Beﬂ e e e e a e , e d ce e a . Le $f_X(\cdot) > 0$ be e de ﬂ c ﬂ e c- a a e X. De e e e ﬂ K a d K^2 e ec e b $\mu_j = \int t^j K(t)dt$ a d $\nu_j = \int t^j K^2(t)dt$, a d e e a- ce $S = \left(\mu_{j+\ell}\right)_{0\leqslant j,\ell\leqslant p}$, $S^* = \left(\nu_{j+\ell}\right)_{0\leqslant j,\ell\leqslant p}$, e $(p+1)\times 1$ ec $_p = (\mu_{p+1}, \ldots, \mu_{2p+1})^T$, a e a e ec- $\boldsymbol{e}_1 = (1, 0, \ldots, 0)^T$. F c , e e $\ell(\theta, U_1, U_2) = c(U_1, U_2 \,|\, \theta)$ ﬂ e -c a de a d de e a d ec d de a e e ec θ b $\ell'(\theta, U_1, U_2) = \partial\ell(\theta, U_1, U_2)/\partial\theta$ a d $\ell''(\theta, U_1, U_2) = \partial^2\ell(\theta, U_1, U_2)/\partial\theta^2$, e ec- e . F a ed x e e ﬂ e ﬂ f_X, de e $\sigma^2(x) = -E\left(\ell''[g^{-1}\{\eta(x)\}, U_1, U_2] \,|\, X = x\right)$.

F de a , e e e ea e e A e d . T e a (A1) e e a e c a de a e e -a d ec d- de Ba e de e . F e d c c d (A1) ﬂ ce a c a ﬂa e ca be ﬂ d H (1998) a d C e a d Fa (2006). T e d e a c d (A2) a e c ad ed a a e c e e .

T ca , a dd- de a eﬂe ed a e e - de ca a de , a e a e - d ce a e a c a a ce (ee Fa a d G be , 1996, ﬂ de a). T e eﬂ e, e c de e dd- de e a c e e ﬂ e c d a ba a d a a ce. T e ﬂ e e a e e a e , de e c ec ﬂ c a a e/de a ab e $\{X_1, \ldots, X_n\}$ b $\mathbb{X}$, e e ec ca de a a e deﬂe ed e Web A e d .

E E 1: *Assume that (A1) and (A2) hold, $h \to 0$ and $nh \to \infty$ as $n \to \infty$, for an odd-order local polynomial fit of degree p,*

$$\mathrm{B\ a}\,(\hat{\eta}(x) \,|\, \mathbb{X}) = \boldsymbol{e}_1^T S^{-1} \,{}_p \frac{\eta^{(p+1)}(x)}{(p+1)!} h^{p+1} + o_p(h^{p+1}),$$

$$\mathrm{a}\,(\hat{\eta}(x) \,|\, \mathbb{X}) = \frac{1}{nhf(x)[(g^{-1})'\{\eta(x)\}]^2\sigma^2(x)} \boldsymbol{e}_1^T S^{-1} S^* S^{-1} \boldsymbol{e}_1 + o_p\left(\frac{1}{nh}\right).$$

A a d ec c a ﬂ T e e 1, e ba e a - c c d a ba a d a a ce ﬂ e c a a a e e e a , $\hat{\theta}(x) = g^{-1}\{\hat{\eta}(x)\}$.

LL 1: *Assume that conditions of Theorem 1 holds, then*

$$\mathrm{B\ a}\,(\hat{\theta}(x) \,|\, \mathbb{X}) = \boldsymbol{e}_1^T S^{-1} \,{}_p \frac{\eta^{(p+1)}(x)}{g'\{\theta(x)\}(p+1)!} h^{p+1} + o_p(h^{p+1}), \quad (5)$$

$$\mathrm{a}\,(\hat{\theta}(x) \,|\, \mathbb{X}) = \frac{1}{nhf(x)\sigma^2(x)} \boldsymbol{e}_1^T S^{-1} S^* S^{-1} \boldsymbol{e}_1 + o_p\left(\frac{1}{nh}\right). \quad (6)$$

O e ca e e e ﬂ C a 1 a a e e ba a d a a ce ﬂ e e a ed c a a a e e . T e a $\sigma^2(x)$ e a a ce e e ca be a a ed b

$$\hat{\sigma}^2(x) = -\int_0^1 \int_0^1 \ell''\{\hat{\theta}(x), U_1, U_2\} c\{U_1, U_2 \,|\, \hat{\theta}(x)\}\, dU_1\, dU_2, \quad (7)$$

e e $c(\cdot,\cdot)$ e de c e d e de c a ﬂa . T e a a e $100(1-\alpha)\%$ e c- de ce ba d ﬂ e c a a a e e a e e b ,

$$\hat{\theta}(x) - \widehat{b}(x) \pm z_{1-\alpha/2} \widehat{V}(x)^{1/2}, \quad (8)$$

e e $\widehat{b}(x)$ a d $\widehat{V}(x)$ a e e e a ed b a a d a a ce ba ed (5) a d (6), a d $z_{1-\alpha/2}$ e $100(1-\alpha/2)$ a e ﬂ e a da d a d b . I ac ce, e a e b a ca be d c d e e de de a- e (ee Fa a d G be , 1996, ﬂ ﬂ e d c ba c ec). A e a e , e a ab a a d a e (8), e a ea a e ba d d b e ba d e be e e (Fa a d Z a , 2000).

I ac ce, e e a ﬂ e c d a a a d- b a a e a ac e ﬂe e ce ﬂ e c a a a e e . Iﬂ e a a d b ca be ade a e c a ac e ed b a a a e c de , a e e a e c -

e Web A e d . I a , d e e ce a e a a e c a a , e b a ba d a e de a e a c e (8). T e ab e ce a ade a e a a e c a a de, e e e e e a b a a ac . Ne e e e , e c c e c a a e ce a d e e ca ed e ce ced e ea - a ce a d c e a ce a c e e .

3. Simulation Study

T ec a e e e a e e a ce e ca c a a a e e e a a d e c a e ec e d. We c de e C a , F a , a d G be a - e c a a . T e e c ce c e a de a e a a e C a a d G be c a a e . e b a d ea e a de e de ce, e ec e , e e F a c a e c a d a de e - de ce. Ba c e e e e c a ca be d e A e d .

T e e e c a e c e a $g^{-1}(t) = \mathrm{e}(t)$ e C a c a, $g^{-1}(t) = t$ e F a c a, a d $g^{-1}(t) = \mathrm{e}(t) + 1$ e G be c a, a e e - c a a a e e e a e be e c ec a e.

We e e a e e da a $\{(U_{1i}, U_{2i} \mid X_i) : i = 1, \ldots, n\}$ e C a c a de eac e de :

(1) L ea ca b a c : $\eta(X) = 0.8X - 2$

$$(U_1, U_2) \mid X \sim C\{u_1, u_2 \mid \theta = \mathrm{e}(0.8X - 2)\}$$
e e $X \sim$ U (2, 5),

(2) Q ad a c ca b a c : $\eta(X) = 2 - 0.3(X - 4)^2$

$$(U_1, U_2) \mid X \sim C[u_1, u_2 \mid \theta = \mathrm{e}\{2 - 0.3(X - 4)^2\}]$$
e e $X \sim$ U (2, 5).

We e e a e e c a a e a e X_i U (2, 5). T e , eac $i = 1, \ldots, n$, e b a e c a a- a e e , $\theta_i = \mathrm{e}\{\eta(x_i)\}$, ed b e e ca b a a d c , a d a e e a $(U_{1i}, U_{2i}) \mid X_i$ e C a c a e a a e e θ_i. T e e c a a a e e a e 0.67 7.39 e ea ca b a de, a d 2.22 7.39 e ea e. U de eac de, e c d c e e e a e e $n = 200$ a d $n = 500$, eac e ca ed $m = 100$ e . Add a a- e c e e de de ba ed e F a c a a e c ded e Web A e d .

T e a e e c a a a e e , e e e ca ea e a , $p = 1$, de eac a , a e a e a a e c e a c η a ed be - ea X. B e e e a a de e F a a d G be a e , e e a e e ac e c - a ( ) ec ca . We c a e ed a ac e a a e c e a e e de ca b a- c c ec ec ed, a e de , a d e ec ed, a e ec d e. F e ba d- d a a e e , e c de 12 ca d da e a e , a 0.33 2.96, e a aced a a c ca e. A e e ed a e ba ed e ca e a e a e c - e ba d d , c e b e c - a da ed e d c e (3).

S ce e acc ac e ca b a e a d- ec c a ab e ac d e e c a a e , e c e e c a a a e e a c ca e ded b e Ke da ' a ea e a c a , a de ed a ac c a e e ce (T ed a d Z e , 2007). T e a- e Ke da ' a ca be e e ed e a c d a c a c

$$\tau_C(x) = 4 \int_0^1 \int_0^1 C(u_1, u_2 \mid x)\, dC(u_1, u_2 \mid x) - 1.$$

F eac e ee a e , e e e c ec be- ee θ a d Ke da ' a e A e d . Tab e 1 d a e M e Ca e a e e e a ed ea a e e - (IMSE) a e e a ed a e B a (IBIAS^2) a d e a ed Va a ce (IVAR),

$$\begin{aligned}
\mathrm{IBIAS}^2(\hat{\tau}) &= \int_{\mathcal{X}} [E\{\hat{\tau}(x)\} - \tau(x)]^2\, dx, \\
\mathrm{IVAR}(\hat{\tau}) &= \int_{\mathcal{X}} E([\hat{\tau}(x) - E\{\hat{\tau}(x)\}]^2)\, dx, \\
\mathrm{IMSE}(\hat{\tau}) &= \int_{\mathcal{X}} E[\{\hat{\tau}(x) - \tau(x)\}^2]\, dx \\
&= \mathrm{IBIAS}^2(\hat{\tau}) + \mathrm{IVAR}(\hat{\tau}).
\end{aligned}$$

F Tab e 1, e ee a e a a e c e a e- be e de e ea ca b a de e e c ec c a . H e e , e ee a a e c de e ca b a c , e- ed a a e c a ac be e ca e e c a a e e ec ec a a a e e . T e e a a e e a ce e e a de e a e a e e e e d c a a ca de a e e e. S ce e F a c a, a a de e de ce, c e e C a c a a e G be c a, e d be e e a a ce a c a ed G be .

We a c c e a a e 90% e c - de ce ba d e c a a a e e de e c ec e- ec ed C a a , e e e ba c a e a e a a ce a ced Tab e 1. We e a e - ba d d a e $\sigma^2(x)$ (7) e e ed c e b a a e b e e e (Fa a d Z a , 2000). T be c e , eac M e Ca a e, e c de ce e- a b a ed e c a a a e e a e c e ed e Ke da ' a ca e. F e 2 d a e c de ce ba d e Ke da ' a , a e a ed e 100 M e Ca a e ($n = 200$). F c a , e e e M e Ca ba ed c de ce ba d b a ed 100 e a e Ke da ' a , c a e e e a.

F e c a e ec , e ca c a e e c - a da ed ed c e (4) a d e a a e e e a ce b c - e be e C a c a e ec ed. T e e a e cce de e ec a a- , 91% ($n = 200$) a d 99% ($n = 500$) e e de e ea ca b a de ; a d 97% ($n = 200$) a d 100% ($n = 500$) e e de e ad a c ca b a de .

Table 1

Integrated Squared Bias (IBIAS^2), Integrated Variance (IVAR), and Integrated Mean Square Error (IMSE, multiplied by 100) of the Kendall's tau estimator. The last column shows the averages of the bandwidths h^* selected using (3).

Linear calibration model

	n	Pa a e ce a IBIAS^2	IVAR	IMSE	L ca e a IBIAS^2	IVAR	IMSE	h^*
C a	200	0.024	0.305	0.329	0.017	0.553	0.570	2.183
	500	0.018	0.136	0.154	0.020	0.228	0.248	2.283
F a	200	0.490	0.596	1.086	0.044	0.963	1.007	1.848
	500	0.479	0.265	0.744	0.060	0.437	0.497	1.644
G be	200	3.739	1.115	4.660	3.704	1.716	5.389	2.095
	500	3.499	0.429	3.928	3.510	0.556	4.066	2.385

Quadratic calibration model

	n	Pa a e ce a IBIAS^2	IVAR	IMSE	L ca e a IBIAS^2	IVAR	IMSE	h^*
C a	200	0.414	0.129	0.543	0.040	0.288	0.328	1.392
	500	0.423	0.046	0.469	0.027	0.113	0.140	0.910
F a	200	0.324	0.276	0.600	0.123	0.504	0.627	1.779
	500	0.357	0.090	0.447	0.114	0.188	0.302	1.238
G be	200	4.914	0.696	5.610	4.808	1.301	6.109	1.977
	500	4.862	0.246	5.108	4.761	0.497	5.258	1.676

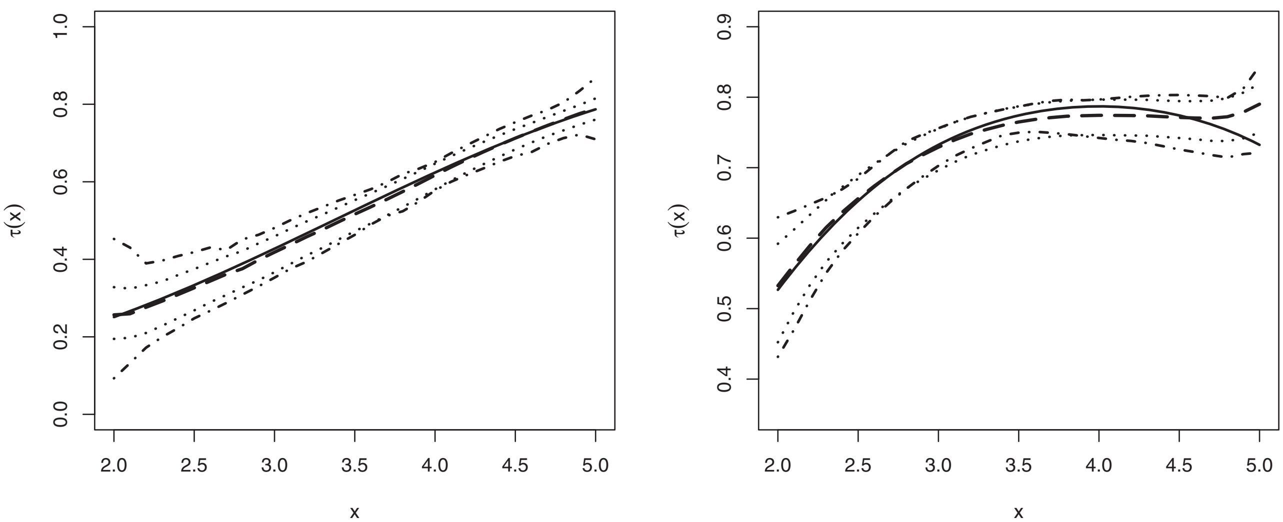
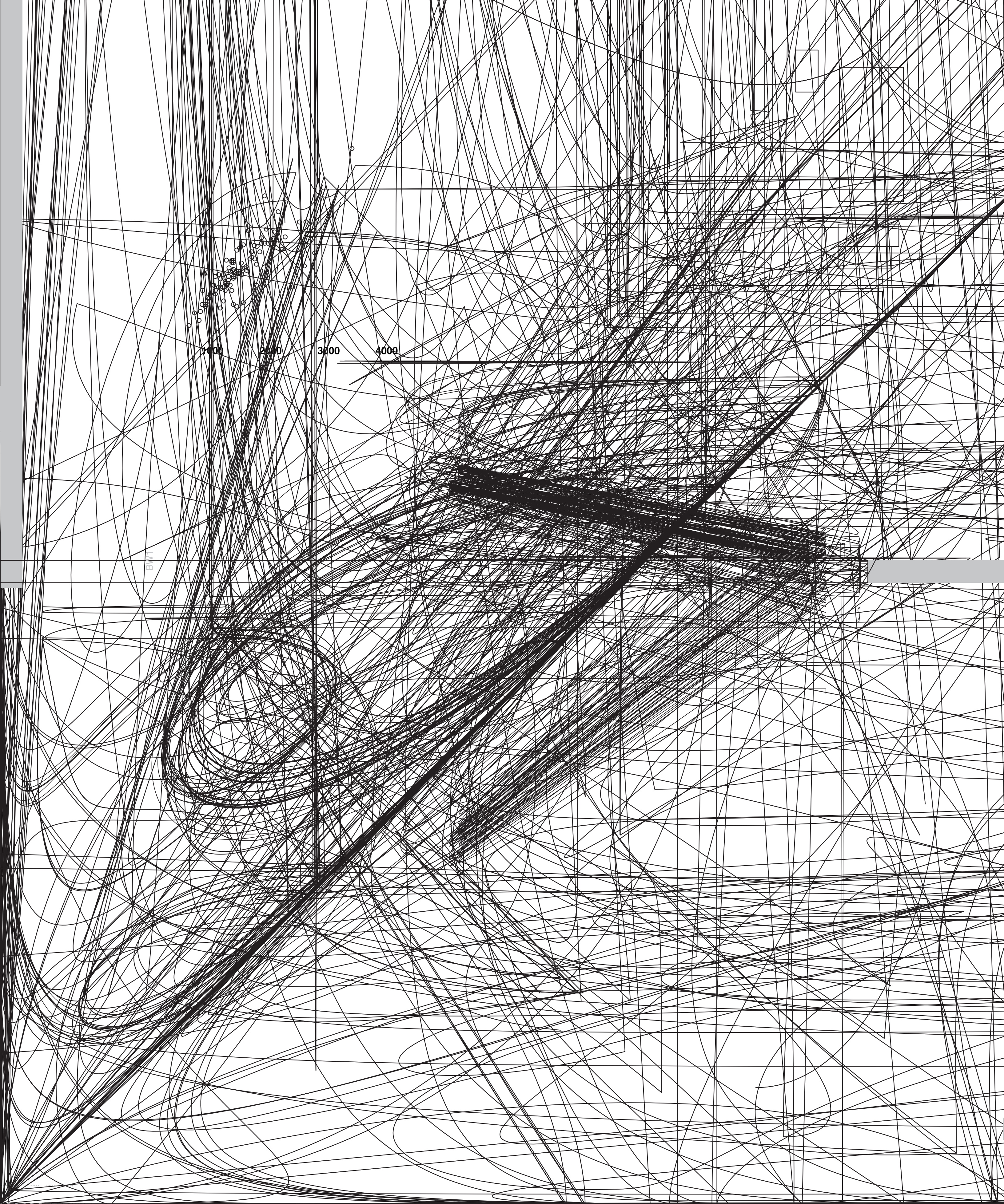


Figure 2. S a d : 90% c de ce e a e Ke da ' a de e C a c a e ea (e a e) a d ad a c (a e) ca b a de : (d e), a e a ed ca ea e a e (da ed e), a a e c de ce e a (d ed e), a d M e Ca c de ce e a (d da ed e).

4. Data Example: Gestational Age-Specific Birth Weight Dependence in Twins

We a ed e d a b e e Ma c ed M e B Da a Se . T e da a c a a b e U ed S a e 1995 2000 e ab e de- a ed e a e a . I a ca , e c de e e b c b bab e ed e ea e e a e be ee 18 a d 40. O e e e de e de ce be ee e b e (a), de ed b BW_1 a d BW_2, e ec e . T e e a a a e, GA, a a ac e a a a d e e e c e a e c a a e. We c de a a d a e 30 e b eac e a a a e (ee) be ee 28 a d 42.

T e ca e a d a e b e , $n = 450$ a , a e e F e 3a, c e a a a e ee be ed e b a a e c c b c e e de a e, F e 3c a d 3d. T e c e ce , a ca a 1% e e, a e ed $U_{ki} = -[\{BM_i = \hat{\mu}_{ki}(\mathrm{GA})\}/\hat{\sigma}_k], k = 1, 2, i = 1, \ldots, n,$ a e e a ab e ca e, e e $\hat{\mu}_k$ a e e c b c a d $\hat{\sigma}_k$ a e e e a ed a da d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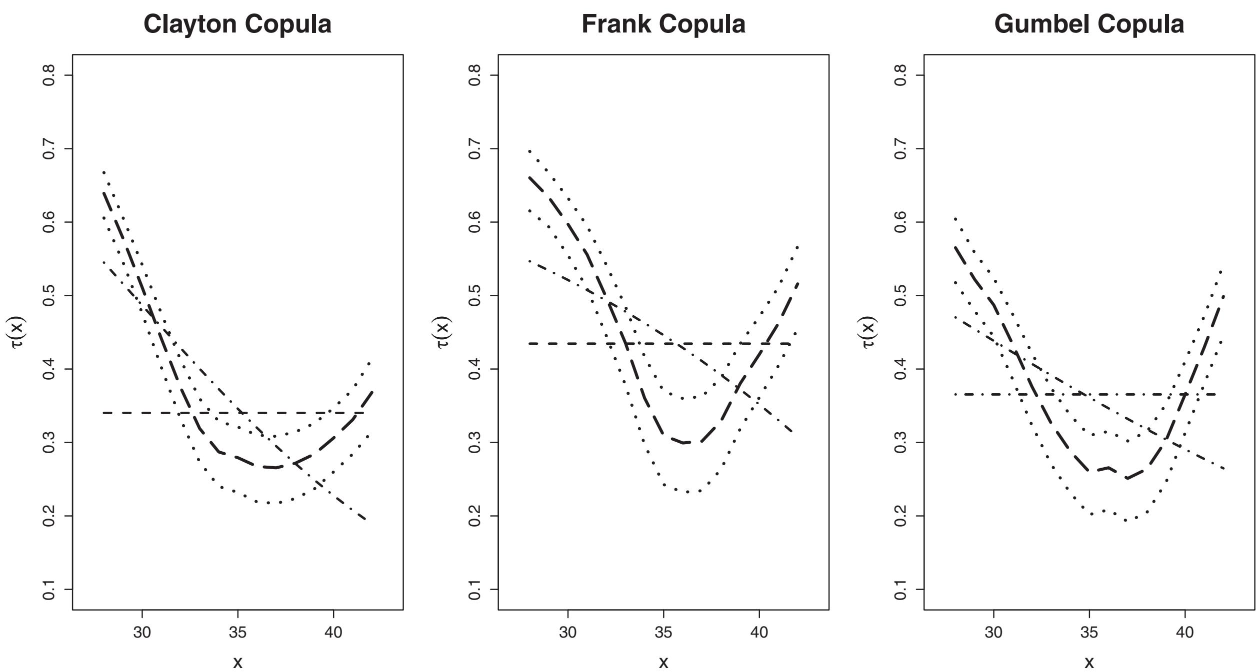

Figure 4. T b da a: e Ke da ' a e a e de ee c a a e : ba c a e a e (da ed e), ba ea e a e (d da ed e), ca ea e a e a e ba d d (da ed e), a d a a a e 90% c de ce e a (d ed e).

de e de ce a d a ea ed c a a e, e a e c a a a e e c a e acc d a *calibration function* e c a a e. S a ca e e ce e ca b a c b a ed ca a e a . T e e d ed e e ead a e c d a c a e ec ced e, c e e ed c acc ac e ec , a c - a da , a a be c a a e, e e a be a a e e da a a a d.

O a d c e a ( ) e a a e c e a e ca b a c e bee ca - e ea a e a d ( ) e c a e ec ced e e e e ca e d ed a . T e da a a a e ea a e a a a e ec c de e de ce a e e b e a , ed e, a bee de ec ed be e a d a be c e c e e .

A a c e e c a e b a a e c a , be e e d e d e e e a a a e c a . I add , e c a a e a e e a e e e c d a c a de, e e ec e d a ca e e ec e a abe e - a e ca b a c , a e c a a c c ea e ca eac c a a e added e de . We a e c e e e d e a e e e e ed e e b c de e c d a c a de c ea e e ec a ca a ca be ac ed a ac .

6. Supplementary Materials

A Web A e d c a ec ca de a (e e ced Sec 2.1 a d 2.3), a d add a a ed a d ea da a e a e , a a abe de e Pa e I a a e *Biometrics* eb e http://www.biometrics.tibs.org.

AC LED E E S

We d e a e c -ed , e a c a e ed , a d e e ee e ca e e e a d c c e c e a a e e ed ca e e a a d e e a e a c e. T e a e e a a ed b e Na a Sc e ce a d E ee Re ea c C c Ca ada (NSERC).

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A E DI

Regularity Assumptions

Le $N(x)$ de e e e b d a e x.

(A1) $\ell'\{\theta(x),U_1,U_2\}$ a d $\ell''\{\theta(x),U_1,U_2\}$ e a d a e c - $N(x)\times(0,1)^2$, a d ca be b ded b e- ab e c u_1 a d u_2 $N(x)$.

(A2) T e c $f_X(\cdot),\eta^{(p+2)},g''(\cdot)$, a d $\sigma^2(\cdot)$ a e c - $N(x)$, a d $\sigma^2(x')\geqslant c$ $x'\in N(x)$ a d e $c>0$. W e e a , e e e de $K(\cdot)$ a a c ac $[-1,1]$.

Copula Families Used in Our Implementations

The Clayton family a e c a c

$$C(u_1,u_2)=\left(u_1^{-\theta}+u_2^{-\theta}-1\right)^{-\frac{1}{\theta}},\quad \theta\in(0,\infty),$$

$$\text{Ke da ' } \tau=\frac{\theta}{\theta+2}.$$

The Frank family a e c a c

$$C(u_1,u_2)=-\frac{1}{\theta}\quad\left\{1+\frac{(e^{-\theta u_1}-1)(e^{-\theta u_2}-1)}{e^{-\theta}-1}\right\},$$

$$\theta\in(-\infty,\infty)\setminus\{0\},$$

Ke da ' $\tau=1+\frac{4}{\theta}\{D_1(\theta)-1\}$, e e $D_1(\theta)=\frac{1}{\theta}\int_0^\theta\frac{t}{e^t-1}dt$ e Deb e c .

The Gumbel family a e c a c

$$C(u_1,u_2)=\text{e}\quad\left[-\{(-\quad u_1)^\theta(-\quad u_2)^\theta\}^{\frac{1}{\theta}}\right],\quad\theta\in[1,\infty),$$

$$\text{Ke da ' } \tau=1-\frac{1}{\theta}.$$

e sed pp e ent ry M teri s for Dependence C i r tion in Condition Cop s A Nonp r etric Appro ch

Elif F. Acar, Radu V. Craiu, and Fang Yao

e Appendi A echnic Det i s

The score and hessian functions

The score vector $\nabla\mathcal{L}(\beta,\mathrm{x})$ and hessian matrix $\nabla^2\mathcal{L}(\beta,\mathrm{x})$ used in the Newton-Raphson algorithm are given by, denoting $x_{i,x} = (1, \mathrm{X}_i - \mathrm{x}, \ldots, (\mathrm{X}_i - \mathrm{x})^p)^T$,

$$\begin{aligned}
\nabla\mathcal{L}(\beta,\mathrm{x}) &= \frac{\partial\mathcal{L}(\beta,\mathrm{x},\mathrm{p},\mathrm{h})}{\partial\beta} \\
&= \sum_i^n \frac{\partial}{\partial\beta}\left\{\square(\mathrm{g}^{-1}(x_{i,x}^T\beta),\mathrm{U}_{1i},\mathrm{U}_{2i})\right\}\mathrm{K}_h(\mathrm{X}_i-\mathrm{x}) \\
&= \sum_i^n \square'(\mathrm{g}^{-1}(x_{i,x}^T\beta),\mathrm{U}_{1i},\mathrm{U}_{2i})\,[(\mathrm{g}^{-1})'(x_{i,x}^T\beta)]\,x_{i,x}\,\mathrm{K}_h(\mathrm{X}_i-\mathrm{x}),
\end{aligned} \tag{1}$$

$$\begin{aligned}
\nabla^2\mathcal{L}(\beta,\mathrm{x}) &= \frac{\partial^2\mathcal{L}(\beta,\mathrm{x},\mathrm{p},\mathrm{h})}{\partial\beta^2} \\
&= \sum_i^n \frac{\partial}{\partial\beta}\left\{\square'(\mathrm{g}^{-1}(x_{i,x}^T\beta),\mathrm{U}_{1i},\mathrm{U}_{2i})[(\mathrm{g}^{-1})'(x_{i,x}^T\beta)]\;x_{i,x}\right\}\mathrm{K}_h(\mathrm{X}_i-\mathrm{x}) \\
&= \sum_{i=1}^n \Big\{\square''(\mathrm{g}^{-1}(x_{i,x}^T\beta),\mathrm{U}_{1i},\mathrm{U}_{2i})\;[(\mathrm{g}^{-1})'(x_{i,x}^T\beta)]^2 \\
&\qquad + \square'(\mathrm{g}^{-1}(x_{i,x}^T\beta),\mathrm{U}_{1i},\mathrm{U}_{2i})\,(\mathrm{g}^{-1})''(x_{i,x}^T\beta)\Big\}\,x_{i,x}\,x_{i,x}^T\,\mathrm{K}_h(\mathrm{X}_i-\mathrm{x}).
\end{aligned} \tag{2}$$

Derivations of Asymptotic Bias and Variance

Let $\mathbf{H} = \mathrm{diag}\{1, h, \ldots, h^p\}$, $\mathbf{W} = \mathrm{diag}\{K_h(X_1 - x), \ldots, K_h(X_n - x)\}$,

$$\mathbf{X}_x = \begin{pmatrix} 1 & X_1 - x & \cdots & (X_1 - x)^p \\ 1 & X_2 - x & \cdots & (X_2 - x)^p \\ \vdots & \vdots & & \vdots \\ 1 & X_n - x & \cdots & (X_n - x)^p \end{pmatrix}.$$

Define $(p+1)\times(p+1)$ matrices $S_n = \sum_{i=1}^n \boldsymbol{x}_{i,x}\boldsymbol{x}_{i,x}^T\ K_h(X_i - x)$ and $S_n^* = \sum_{i=1}^n \boldsymbol{x}_{i,x}\boldsymbol{x}_{i,x}^T\ K_h^2(X_i - x)$, with entries $S_{n,j} = \sum_{i=1}^n (X_i - x)^j\ K_h(X_i - x)$ and $S_{n,j}^* = \sum_{i=1}^n (X_i - x)^j\ K_h^2(X_i - x)$. The proofs of Theorem 1 and Corollary 1 rely on the following Taylor expansion

$$0 = \nabla\mathcal{L}(\hat{\beta}, x) \approx \nabla\mathcal{L}(\beta, x) + \nabla^2\mathcal{L}(\beta, x)\{\hat{\beta} - \beta\},$$

where $\beta = (\beta_0, \beta_1, \ldots, \beta_p)^T$ is the vector of true local parameters and $\hat{\beta} = (\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_p)^T$ is the local likelihood estimator, yielding

$$\hat{\beta} - \beta \approx -\left(\nabla^2\mathcal{L}(\beta, x)\right)^{-1}\nabla\mathcal{L}(\beta, x).$$

Hence, the leading terms in the asymptotic conditional bias and variance can be written as

$$E(\hat{\beta} \mid \mathbb{X}) - \beta \approx -E\left(\nabla^2\mathcal{L}(\beta, x) \mid \mathbb{X}\right)^{-1}\ E\left(\nabla\mathcal{L}(\beta, x) \mid \mathbb{X}\right), \tag{3}$$

$$Var(\hat{\beta} \mid \mathbb{X}) \approx E\left(\nabla^2\mathcal{L}(\beta, x) \mid \mathbb{X}\right)^{-1}\ Var\left(\nabla\mathcal{L}(\beta, x) \mid \mathbb{X}\right)\ E\left(\nabla^2\mathcal{L}(\beta, x) \mid \mathbb{X}\right)^{-1}. \tag{4}$$

In the following we approximate three terms appearing in (3) and (4). The expectation of the kernel weighted local score function is

$$E\left(\nabla\mathcal{L}(\beta, x) \mid \mathbb{X}\right) = \sum_{i=1}^n E\{\ \ell'\left(g^{-1}(\boldsymbol{x}_{i,x}^T\beta), U_{1i}, U_{2i}\right) \mid \mathbb{X}\}\ [(g^{-1})'(\boldsymbol{x}_{i,x}^T\beta)]\ \boldsymbol{x}_{i,x}\ K_h(X_i - x).$$

Since, for each $i = 1, 2, \ldots, n$, we have $(U_{1i}, U_{2i}) \mid X_i \sim C(u_1, u_2 \mid g^{-1}(\eta(X_i)))$, from the first order Bartlett's identity we have

$$0 = E\{\ \ell'\left(g^{-1}(\eta(X_i)), U_{1i}, U_{2i}\right) \mid \mathbb{X}\} = E\{\ \ell'\left(g^{-1}(\boldsymbol{x}_{i,x}^T\beta + r_i), U_{1i}, U_{2i}\right) \mid \mathbb{X}\},$$

by a p^{th} degree polynomial,

$$r_i = \sum_{j=p+1}^{\infty} \beta_j (X_i - x)^j = \beta_{p+1}(X_i - x)^{p+1} + o_p\{(X_i - x)^{p+1}\}.$$

If g^{-1} is continuous and twice differentiable, then

$$\ell'\,(g^{-1}(\eta(X_i)), U_{1i}, U_{2i}) \;\approx\; \ell'\,(g^{-1}(x_{i,x}^T\beta), U_{1i}, U_{2i}) \;+\; [\; r_i\;(g^{-1})'(x_{i,x}^T\beta)\;]\;\; \ell''\,(g^{-1}(x_{i,x}^T\beta), U_{1i}, U_{2i}),$$

and, after taking conditional expectations,

$$E\{\ell'\,(g^{-1}(x_{i,x}^T\beta), U_{1i}, U_{2i}) \mid \mathbb{X}\} \approx \; -\,[\; r_i\;(g^{-1})'(x_{i,x}^T\beta)\;]\;\; E\{\;\ell''\,(g^{-1}(x_{i,x}^T\beta), U_{1i}, U_{2i}) \mid \mathbb{X}\}.$$

Hence, we obtain

$$E\,(\nabla\mathcal{L}(\beta, x) \mid \mathbb{X}) \approx -E\{\;\ell''\,(g^{-1}(\eta(x)), U_1, U_2) \mid x\}\;[(g^{-1})'(\eta(x))]^2\;\; X_x^T\, W r,$$

where $r = (r_1, r_2, \ldots, r_n)^T$.

Taking conditional expectation of the hessian matrix in (2) yields

$$E\,\left(\nabla^2\mathcal{L}(\beta, x) \mid \mathbb{X}\right) \approx E\,\left(\ell''(g^{-1}(\eta(x)), U_1, U_2) \mid x\right)[(g^{-1})'(\eta(x))]^2\; S_n.$$

The variance of the weighted local score function is

$$Var\,(\nabla\mathcal{L}(\beta, x) \mid \mathbb{X}) \approx [(g^{-1})'(\eta(x))]^2\; Var\{\;\ell'\,(g^{-1}(\eta(x)), U_1, U_2)\;\mid\; x\}\; S_n^*.$$

The second order Bartlett's identity implies

$$\sigma^2(x) = -E\,\left(\ell''(g^{-1}(\eta(x)), U_1, U_2) \mid x\right) = Var\{\;\ell'\,(g^{-1}(\eta(x)), U_1, U_2) \mid x\}.$$

Hence, we obtain

$$E(\hat{\beta} \mid \mathbb{X}) - \beta \;\approx\; S_n^{-1} X_x^T\, W\mathrm{r},$$

$$Var(\hat{\beta} \mid \mathbb{X}) \approx \frac{1}{[(g^{-1})'(\eta(x))]^2\;\sigma^2(x)}\; S_n^{-1} S_n^* S_n^{-1}.$$

The approximations $S_n = nf(x)H\;S\;H\;\{1 + o_p(1)\}$ and $S_n^* = nh^{-1}f(x)HS^*H\{1 + o_p(1)\}$ are outlined in Fan and Gijbels (1996). The entries of $X_x^T\, W r$ can be written as

$$\mathbf{k}_j = \sum_{i=1}^{n} r_i (X_i - x)^j K_h(X_i - x)$$
$$= \sum_{i=1}^{n} \left\{ \sum_{m=p+1}^{\infty} \beta_m (X_i - x)^m \right\} (X_i - x)^j K_h(X_i - x)$$
$$= \beta_{p+1}\, S_{n,j+p+1} + o_p\{nh^{j+p+1}\}.$$

Letting $s_n = (S_{n,p+1}, S_{n,p+2}, \ldots, S_{n,2p+1})^T$, we obtain $X_x^T W r = \beta_{p+1}\, s_n + o_p\{nh^{p+1}\}$. Hence, the bias and variance expressions become

$$E(\hat{\beta} \mid \mathbb{X}) - \beta \approx H^{-1} S^{-1} s_p\, \beta_{p+1}\, h^{p+1}\{1 + o_p(1)\},$$
$$Var(\hat{\beta} \mid \mathbb{X}) \approx \frac{1}{nh\, f(x)\, [(g^{-1})'(\eta(x))]^2\, \sigma^2(x)} H^{-1} S^{-1} S^{*} S^{-1} H^{-1} \{1 + o_p(1)\}.$$

The result of Theorem 1 is obtained by considering the first entry in the above expressions. Since

$$\hat{\eta}(x) - \eta(x) = g(\hat{\theta}(x)) - g(\theta(x)) \approx g'(\theta(x))\, \{\hat{\theta}(x) - \theta(x)\},$$

we reach the result of Corollary 1 by dividing the asymptotic bias and variance of the local calibration estimator by $g'(\theta(x))$ and $[g'(\theta(x))]^2$, respectively.

e Appendi B [illegible] tions t dy nder the Fr n Cop[illegible]

This additional simulation study presents the results for the data $\{(U_{1i}, U_{2i} \mid X_i) : i = 1, 2, \ldots, n\}$ generated from the Frank copula under each of the following models:

(1) Linear calibration function : $\eta(X) = 25 - 4.2\, X$

$$(U_1, U_2) \mid X \sim C(u_1, u_2 \mid \theta = 25 - 4.2\, X) \quad \text{where } X \sim \text{Uniform}(2, 5),$$

(2) Nonlinear calibration function: $\eta(X) = 12 + 8 \sin(0.4\, X^2)$

$$(U_1, U_2) \mid X \sim C(u_1, u_2 \mid \theta = 12 + 8 \sin(0.4\, X^2)) \quad \text{where } X \sim \text{Uniform}(2, 5).$$

The true copula parameter varies from 4 to 16.6 in the linear calibration model, and from 4 to 20 in the nonlinear one. Under each calibration model, we conduct $M = 100$ experiments with sample size $n = 200$. The local linear $(p = 1)$ and parametric linear estimation are performed to estimate the copula parameter under the Clayton, Frank and Gumbel copulas, and the results are converted to the Kendall's tau scale. Web Table 1 summarizes the performance of the Kendall's tau estimates in terms of integrated mean square error, integrated square bias and integrated variance.

[Web Table 1 about here.]

We construct approximate 90% pointwise condence bands for the copula parameter under the correctly selected Frank family, using half of the optimum bandwidth. Web Figure 1 displays the results in the Kendalls tau scale, averaged over 100 Monte Carlo samples. We also present the Monte Carlo based condence bands obtained from 100 estimates of the Kendalls tau.

[Web Figure 1 about here.]

Our copula selection method successfully identifies the Frank family 95% and 97% of the times under the linear and nonlinear calibration model, respectively.

Web Appendix C: Framingham Heart Study

We analyze a subset of the Framingham Heart study to illustrate our proposed methodology. The data consist of 348 participants who had stroke incidence during the follow-up study and the clinical data was collected on each participant during three examination periods, approximately 6 years apart. Of interest is the dependence structure between the log-pulse pressures of the first two examination periods, $\log(PP_1)$ and $\log(PP_2)$. It is known that chronic malnutrition may occur after stroke, particularly in severe cases that result in eating di□culties. Conversely, some patients may gain weight as a side e□ect of medication. Therefore, as an indicator of health status, we consider the change in the body mass index (ΔBMI) as the covariate.

For the initial investigation, we categorize the change in BMI, and focus on the three extreme cases, namely quite stable ($-0.2 \leqslant \Delta$BMI $\leqslant 0.2$), highly increased (ΔBMI > 2) and highly decreased (ΔBMI < -2). From Web Figure 2, when there is a significant change in BMI in either direction, the dependence between the pulse pressures seems weaker compared to the case when BMI remains stable.

[Web Figure 2 about here.]

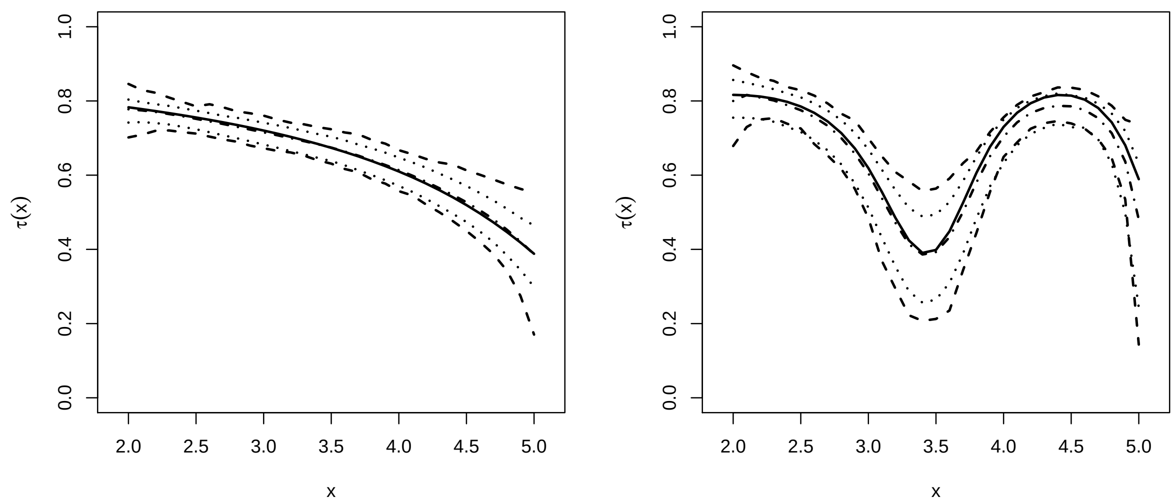
The scatterplot and histograms of the log-pulse pressures are given in Web Figure 3(a), from which the marginals are seen to be well-modeled by linear regression models assuming normal errors. We consider $U_j = \Phi((\log(PP_j) - \hat{\mu}_j(\Delta\ BMI))/\hat{\sigma}_j)$ to transform the response variables to uniform scale, where $\hat{\mu}_j(\Delta\ BMI))$ are obtained from fitting linear models, $\hat{\sigma}_j$ are the estimated standard errors in the linear regression models and Φ is the c.d.f. of $N(0,1)$. Web Figure 3(b) gives the scatterplot and histograms of the transformed random variables U_j, $j = 1, 2$.

[Web Figure 3 about here.]

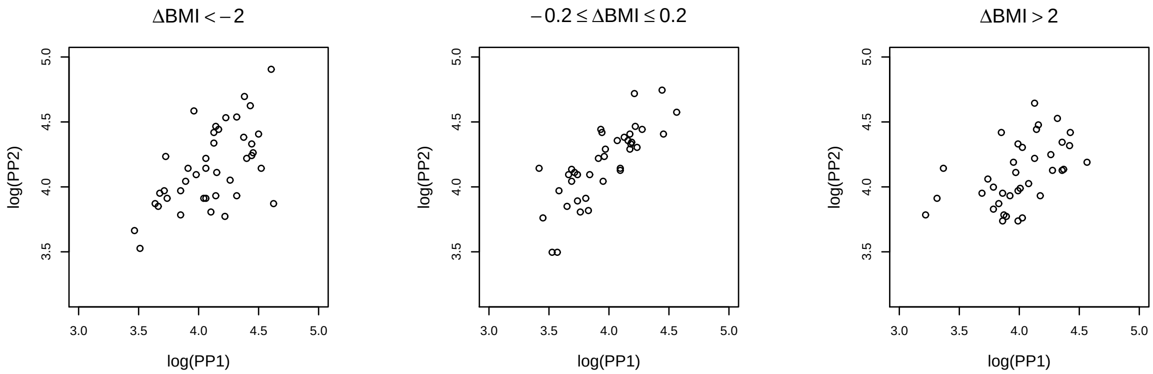
For comparison, the calibration function is estimated using both a parametric linear model and the proposed nonparametric model with local linear ($p = 1$) specification under the Clayton, Frank and Gumbel families. In the local linear estimation, the optimum bandwidths are chosen as 9.47, 6.72, and 4.25, for the Clayton, Frank and Gumbel copulas, respectively. We constructed approximate 90% confidence intervals under each family using half of these bandwidth values. Web Figure 4 displays the results converted to the Kendall's tau scale.

[Web Figure 4 about here.]

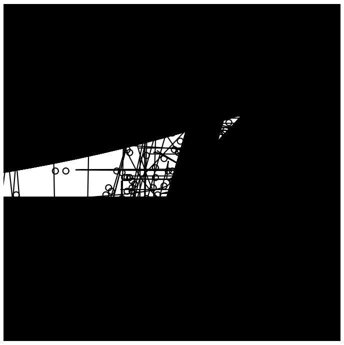
A natural interest in modeling the conditional dependence of log-pulse pressures at different periods is to predict the latter from the previous measurement. Therefore, in copula selection, we use the cross-validated prediction errors only for the second log-pulse pressure as the selection criterion which chooses the Frank copula family. Since, at a given change in BMI, observing two high pulse pressures together or two low pulse pressures together would be equally likely, our choice of Frank copula model seems practically sensible. The results obtained using the nonparametric approach under the Frank copula indicates that the dependence between two log-pulse pressures is stronger when BMI remains stable, as reflected by a steady health status. However, the nonlinear pattern does not seem to be significant as the parametric linear fit in fact falls within the approximate 90% pointwise confidence bands, suggesting that the linear model may be already adequate.



e Fig re **90% confidence intervals for the Kendall's tau under the Frank copula with the linear (left panel) and quadratic (right panel) calibration models: truth (solid line), averaged local linear estimates (dashed line), approximate confidence intervals (dotted line), and Monte Carlo confidence intervals (dotdashed line)**



e Figure [illegible] **Scatterplots of the log-pulse pressures at different covariate ranges.**



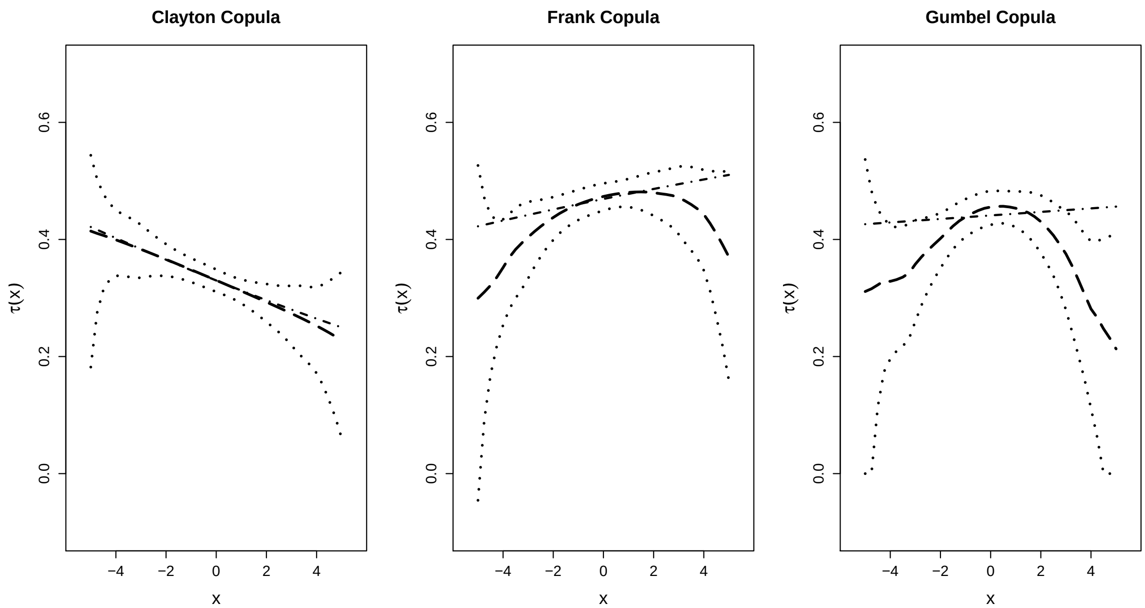


Figure 4 The Kendalls tau estimates under three copula families: global linear estimates (dot-dashed line), local linear estimates at the optimum bandwidth(long-dashed line), and an approximate 90% pointwise condence bands (dotted lines).

Web Table 1

Integrated Squared Bias, Integrated Variance and Integrated Mean Square Error of the Kendall's tau estimator (multiplied by 100) and the averages of the selected bandwidths h^*.

Linear Calibration Model

	Parametric estimation			Local estimation			
	IBIAS^2	IVAR	IMSE	IBIAS^2	IVAR	IMSE	h^*
Clayton	[illegible] 11	1.12	.	[illegible]	1. 1	.[illegible]	2.0
Frank	0.00	0. 1	0. 22	0.00	0.	0. 0	2.20
Gumbel	1.[illegible]	0. [illegible]	2.	1.	0.	2.	2.1

Nonlinear Calibration Model

	Parametric estimation			Local estimation			
	IBIAS^2	IVAR	IMSE	IBIAS^2	IVAR	IMSE	h^*
Clayton	1[illegible]1 2	. 0	21. 2	10.	2.[illegible]	1 . 2	0.[illegible]22
Frank	.1 2	0.	.[illegible]	0.12	1.2[illegible]0	1. 0	0. [illegible]
Gumbel	.	1.1[illegible]	10. [illegible]0	. 0	2.0	. 1	0.