# Functional Additive Models

## 1.

M

*functional additive model* ( M).

M

*Drosophila*

$Y$ $\zeta_m$ $\beta$ $K$ $M$

## 3. A A

$ik$ ( ) ( ) ( ) ( ) M

$$E(Y - \ _Y|\ _k) = b_k\ _k \qquad E(\zeta_m|\ _k) = b_{km}\ _k \quad (\ )$$

M (H ) $b_k\ _k$ $b_{km}\ _k$ ( ) ( ) ( ) $k$ $f_k(\ _k)$ $f_k(\cdot)$ $k = \ , \ , \ldots$ $f_{km}(\cdot)$ $k, m = \ , \ , \ldots$ ( ) ( ) M $k$

$$E(Y|X) = \ _Y + \sum_{k=}^{\infty} f_k(\ _k) \quad (\ )$$

$$E(Y(t)|X) = \ _Y(t) + \sum_{k=}^{\infty} \sum_{m=}^{\infty} f_{km}(\ _k)\ _m(t), \quad (\ )$$

$$Ef_k(\ _k) = \ , \qquad k = \ , \ , \ldots,$$
$$Ef_{km}(\ _k) = \ , \qquad k = \ , \ , \ldots, m = \ , \ , \ldots. \quad (\ )$$

$Y$ $k$ $k$ M ( ) ( )

$$E(Y - \ _Y|\ _k) = E\{E(Y - \ _Y|X)|\ _k\} = E\left\{\sum_{j=}^{\infty} f_j(\ _j)\Big|\ _k\right\} = f_k(\ _k), \quad (\ )$$

$$E(\zeta_m|\ _k) = E\{E(\zeta_m|X)|\ _k\} = E\left\{\sum_{j=}^{\infty} f_{jm}(\ _j)\Big|\ _k\right\} = f_{km}(\ _k). \quad (\ )$$

$f_k(\ _k) = E(Y - \ _Y|\ _k)$ $f_{km}(\ _k) = E(\zeta_m|\ _k)$ $k, m = \ , \ , \ldots,$ ( ) M ( H M ) ( ) ( ) $f_k$ $f_{km}$ M ( ) ( )

## 4. A A

$\hat{\ }_k$ $\hat{\ }_k$ $\hat{\ }_{ik}$ *principal analysis by conditional expectation* ( E ) $K$ ( ) ( ) E

$\hat{f}_k$ $\hat{f}_{km}$ $f_k$ $f_{km}$ $f_k$ $\{\hat{\ }_{ik}, Y_i\}_{i=\ ,\ldots,n}$ $\hat{\ }_{ik}$ ( ) M

$$\sum_{i=}^{n} K\left(\frac{\hat{\ }_{ik} - x}{h_k}\right)\{Y_i - \beta\ - \beta\ (x - \hat{\ }_{ik})\} \quad (\ )$$

$\beta$ $\beta$ $\hat{f}_k(x) = \hat{\beta}\ (x) - \bar{Y}$ $h_k$ $K$ $f_{mk}$

$\{\hat{}_{ik}, \hat{\zeta}_{im}\}_{i=\ ,\dots,n}$ ( )

$$\sum_{i=}^{n} K\left(\frac{\hat{}_{ik}-x}{h_{mk}}\right)\{\hat{\zeta}_{im}-\beta\ -\beta\ (x-\hat{}_{ik})\} \quad (\ )$$

$\beta$ $\beta$ $\hat{f}_{mk}(x)=\hat{\beta}\ (x)$ $h_{mk}$ M ( )

$$\widehat{E}(Y|X)=\bar{Y}+\sum_{k=}^{K}\hat{f}_k(\ _k). \quad (\ )$$

$R$ ( )

$$R = \ -\frac{\sum_{i=}^{n}\{Y_i-E(Y_i|X_i)\}}{\sum_{i=}(Y_i-\ _Y)}$$

$$\widehat{R} = \ -\frac{\sum_{i=}^{n}\{Y_i-\widehat{E}(Y_i|X_i)\}}{\sum_{i=}(Y_i-\bar{Y})}, \quad (\ )$$

$E(Y_i|X_i)$ $\widehat{E}(Y_i|X_i)$ $i$ ( ) ( ).

M ( )

$$\widehat{E}\{Y(t)|X\}=\hat{}_Y(t)+\sum_{m=}^{M}\sum_{k=}^{K}\hat{f}_{mk}(\ _k)\hat{}_m(t), \quad t\in\mathcal{T}, \quad (\ )$$

$$R = \ -\frac{\sum_{i=}^{n}\int\ [Y_i(t)-E\{Y_i(t)|X_i\}]\ dt}{\sum_{i=}^{n}\int\ \{Y_i(t)-\ _Y(t)\}\ dt},$$

$$\widehat{R} = \ -\frac{\sum_{i=}^{n}\sum_{l=}^{m_i}[V_{il}-\widehat{E}\{Y_i(t_{il})|X_i\}]\ (t_{il}-t_{i,l-}\ )}{\sum_{i=}^{n}\sum_{l=}^{m_i}\{V_{il}-\ _Y(t_{il})\}\ (t_{il}-t_{i,l-}\ )} \quad (\ )$$

$E\{Y_i(t)|X_i\}$ $\widehat{E}(Y_i(t)|X_i)$ $i$ ( ) ( ) $t_{il}$.

## 5. A

E $_{ik}$ $\hat{}_{ik}$ $\zeta_{im}$ $\hat{\zeta}_{im}$ $k=\ ,\dots,K$, $m=\ ,\dots,M$. M $f_k$ $f_{km}$ [redacted] ( ) ( )( H H ) [redacted] $|\hat{}_{ik}-\ _{ik}|$ $|\hat{\zeta}_{im}-\zeta_{im}|$ $i$ $i$, $i=\ ,\dots,n$ ( ). $f_k$ ( ) $f_{mk}$ ( )

$\{\hat{}_{ik}, Y_i\}$ $\{\hat{}_{ik}, \hat{\zeta}_{im}\}$ $i=\ ,\dots,n$. $n\to\infty$ $K$ $M$ $K=K(n)\to\infty$ $M=M(n)\to\infty$ ( ) . ( ) ( ) ( ) ( )

*Theorem 1.* U ( ) ( ) ( ) ( ) ( ) ( ) ( ) $k\geq$ $_j, j\leq k$

$$\hat{f}_k(x)-f_k(x)\xrightarrow{p}\ , \quad (\ )$$

( ). U ( ) ( ) ( ) $k$ $m$ $_j, j\leq k$ $_l, l\leq m$

$$\hat{f}_{km}(x)-f_{km}(x)\xrightarrow{p}\ , \quad (\ )$$

( ).

$\tilde{}_k(x)=|\hat{f}_k(x)-f_k(x)|$ $\vartheta_{mk}(x)=|\hat{f}_{mk}(x)-f_{mk}(x)|$ ( ). M.

*Theorem 2.* U ( ) ( ) ( ) ( ) ( ) ( ) ( ) ( )

$$\widehat{E}(Y|X)-E(Y|X)\xrightarrow{p}\ , \quad (\ )$$

$\widehat{E}(Y|X)=\bar{Y}+\sum_{k=}^{K}\hat{f}_k(\ _k)$ ( ). U ( ) ( ) ( ) ( ) ( ) $t\in\mathcal{T}$,

$$\widehat{E}\{Y(t)|X\}-E\{Y(t)|X\}\xrightarrow{p}\ , \quad (\ )$$

$\widehat{E}\{Y(t)|X\}=\hat{}_Y(t)+\sum_{k=}^{K}\sum_{m=}^{M}\hat{f}_{mk}(\ _k)\hat{}_m(t)$ $\hat{f}_{mk}(\ _k)$ ( ).

$^*_n=|\widehat{E}(Y|X)-E(Y|X)|$ $\vartheta^*_n=|\widehat{E}(Y(t)|X)-E(Y(t)|X)|$ ( ).

## 6. A

M ( ) ( ) $n=$ $X_i$ $_X(s)=s+\ (s)$ $\leq s\leq$ $(s)=-\ (\ s/\ )/\sqrt{\ }$ $(s)=\ (\ s/\ )/\sqrt{\ }$ $\leq s\leq$. $=$ $=$ $_k=$ $k\geq$ ( ) $\epsilon_{ij}\sim\ (\ ,.\ )$. ( ) $_{ik}\sim\mathcal{N}(\ ,\ _k)$ ( ) $_{ik}=\sqrt{\ _k}(Z_{ik}-\ )/$ $Z_{ik}\sim$ ( ) [redacted] $k=\ ,\ .$

[ , ]

{ , . . . , }

{ , . . . , }

$X_i^*$ $U_{ij}^*$ $U_{ij}$

( $Y_i^*$)

$Y_i = \sum_{k=} f_k(\ _{ik}) + \varepsilon_i$ $f_k$' $_{ik}$ $X_i$ $Y_i$ $\varepsilon_i \sim$ ( , . ) $_Y =$ $_Y = .$ M ( ) ( ) ( ) $f_k(x) = x - \ _k$ $k =$ , ( ) $f(x) = x$ $f(x) = x/$ ( ) $f_k$ ( ) $\hat{f}_k(x) = \sum_{i=}^n (Y_i - \bar{Y})(\hat{\ }_{ik} - \bar{\hat{\ }}_{\cdot k})\hat{\ }_k^- x$ $\bar{Y}_i = \sum_{i=}^n Y_i/n$ $\bar{\hat{\ }}_{\cdot k} = \sum_{i=}^n \hat{\ }_{ik}/n$ $\hat{\ }_k$ k $X$ $\hat{\ }_{ik}$' E ( )

M

$f_k$

M ( )

$f_{km}$ $\hat{f}_{km}(x) = \sum_{i=}^n (\hat{\zeta}_{im} - \bar{\zeta}_{\cdot m})(\hat{\ }_{ik} - \bar{\hat{\ }}_{\cdot k})\hat{\ }_k^- x$

$Y_i(t) = \ _Y(t) + \zeta_i$ $(t)$ $_Y(t) = t +$ $(t)$ $(t) = -$ ( $t/$ )$/\sqrt{\ }$ $\zeta_i$ $\le t \le$ $\zeta_i$ $f_{km}, k =$ , $, m =$ ( ) $f_k(x) = x -$ $_k$ $k =$ , $\zeta_i = \sum_{k=}(\ _{ik} - \ _k)$ ( ) $f(x) = x$ $f(x) = x/$ $\zeta_i = \ _i + \ _i /$ $\varepsilon_{il}$ ( ) $\mathcal{N}($ , . )

$X_i^*$ $Y_i^*$ $U_{ij}^*$ $V_{il}^*$ $U_{ij}$ $V_{il}$ ( E )

$$\mathrm{E}\ _i = \frac{(Y_i^* - \hat{Y}_i^*)}{Y_i^*}$$

$$\mathrm{E}\ _{i,f} = \frac{\int (Y_i^*(t) - \hat{Y}_i^*(t))\ dt}{\int Y_i^*(t)\,dt}, \qquad ( )$$

E M $f_k$ $f_{km}$ M ( ) M M

M ( ) ( ) M ( ) ( ) M ( )

| | M | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | M | | | | | | | | |
| | M | | | | | | | | |
| | M | | | | | | | | |
| | M | | | | | | | | |

E ( ) ( ) M $n =$ M E ( )

| | M | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | M | | | | | | | |
| | M | | | | | | | |
| | M | | | | | | | |
| | M | | | | | | | |

M

E

M

M

## 7. A A X A A

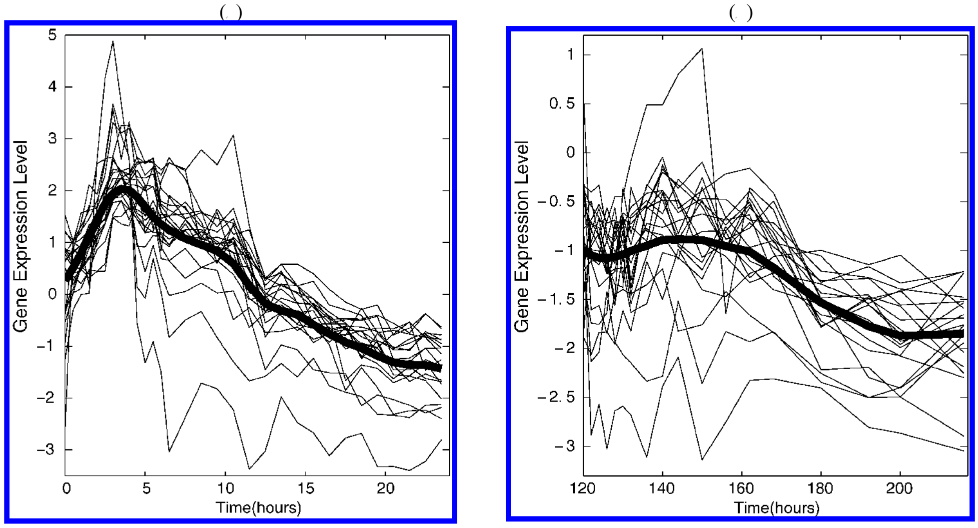
*Drosophila*

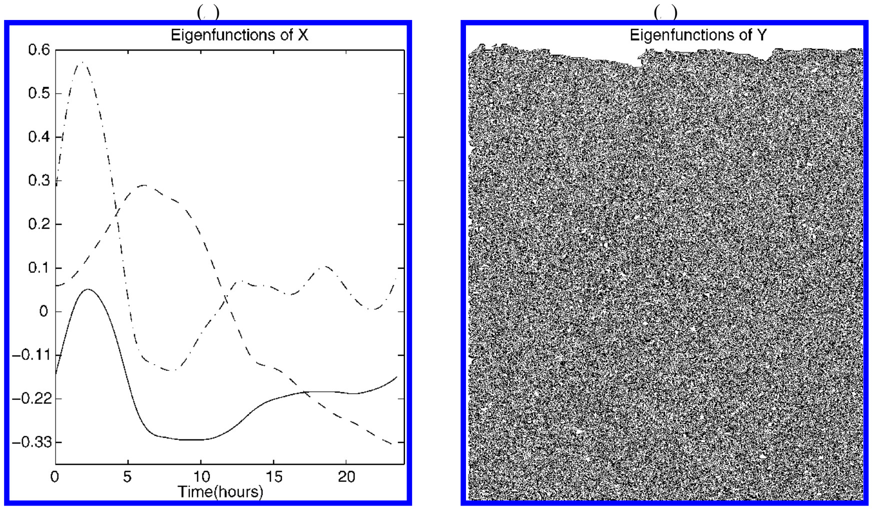
$\hat{\zeta}_i$, $\hat{\zeta}_i$, $\hat{\zeta}_i$, $\hat{\zeta}_i$

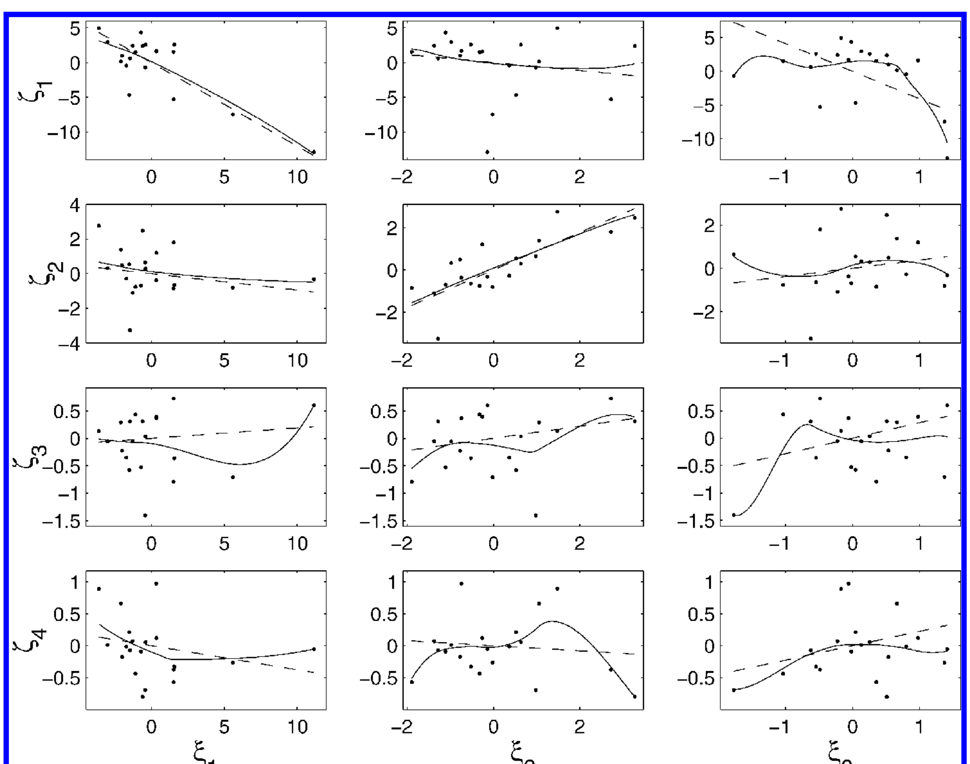
M

$n =$

$\hat{\zeta}_i$




Eigenfunctions of X
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.11
−0.22
−0.33
0
5
10
15
20
Time(hours)
Eigenfunctions of Y


ζ1
ζ2
ζ3
ζ4
ξ1
ξ2
ξ3

$R$ ( )

E $(-i), f$ ( ) M

| | | | M | $R$ |
|---|---|---|---|---|
| M | - | - | - | - - | - |
| | - | - | - | - - | - |

E ( )

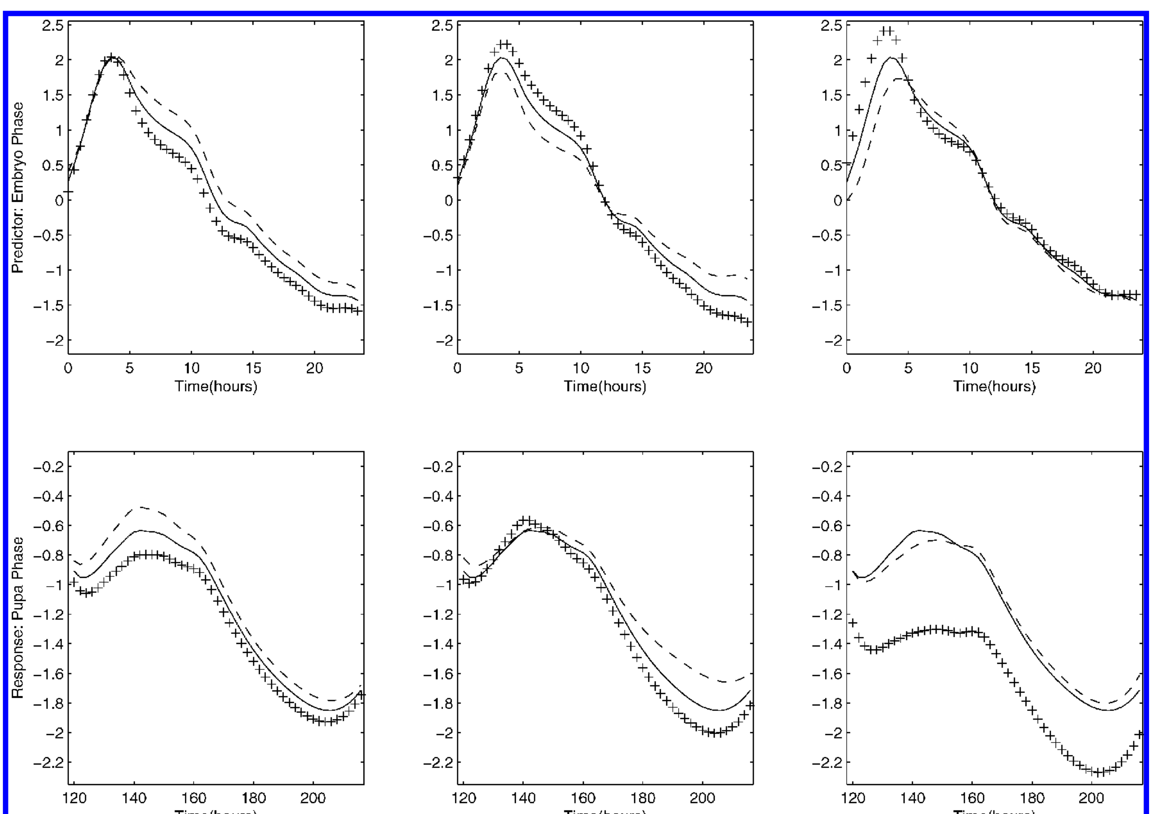
E $(-i), f$

M

$R$ ( ) M

## 8. A

M




$\hat{\mu}_X(s) + \alpha \hat{\phi}_k(s)$ $\alpha =$ ( ) $\alpha =$ (−−) $\alpha = -$ (++) $k =$ ( ) $k =$ ( ) $k =$ ( ) $\hat{\mu}_Y(t) + \sum_{m=}^{M} \{\hat{f}_{km}(\alpha) + \sum_{\ell \neq k} \hat{f}_{\ell m}(\,)\} \hat{\psi}_m(t)$

M

M

A X A

A A

A.1

$k, k =$ , , …,

$X$ $(A_{G_X} f)(s) =$

$f(s)$ $f \in L$ $(\mathcal{S})$,

$(A_{G_X} f)(t) = \int f(s) \times$

$G_X(s,t)\,ds$ H

$\int$ $j(s)$ $k(s)\,ds = \delta_{jk}$

$\delta_{jk} =$ $j = k$ $=$ $j \neq k$. $j$

$\geq$ $\geq \cdots$.

$Y$ $m$ $m$. $\{X, j\}$

$\{Y, k\}$

$G_X(s, s) = \sum_k k \times$

$k(s)$ $k(s)$ $G_Y(t, t) = \sum_m m$ $m(t)$ $(t)$

$X$ $Y$ K

$X$ $Y$

$$X(s) = \quad _X(s) + \sum_{j=}^{\infty} \quad j \quad j(s)$$

$$Y(t) = \quad _Y(t) + \sum_{k=}^{\infty} \zeta_k \quad k(t).$$

$\{k, k = , , \ldots\}$ $\{m, m = , , \ldots\}$

$\beta_k$ $\beta_{km}$ $\beta(s) =$

$\sum_{k=}^{\infty} \beta_k \; k(t)$ $\beta(s,t) = \sum_{k=}^{\infty} \sum_{m=}^{\infty} \beta_{km} \; k(s) \; m(t)$.

( ) ( )

( )

$$U_{ij} = X_i(s_{ij}) + \epsilon_{ij}$$

$$= \quad _X(s_{ij}) + \sum_{k=}^{\infty} \quad ik \quad k(s_{ij}) + \epsilon_{ij},$$

$$s_{ij} \in \mathcal{S}, \quad \leq i \leq n, \quad \leq j \leq n_i,$$

$$V_{il} = Y_i(t_{il}) + \varepsilon_{il}$$

$$= \quad _Y(t_{il}) + \sum_{m=}^{\infty} \zeta_{im} \quad k(t_{il}) + \varepsilon_{il},$$

$$s_{il} \in \mathcal{T}, \quad \leq i \leq n, \quad \leq l \leq m_i.$$

M $\mathbf{U}_i = (U_i, \ldots, U_{in_i})^T$, $X_i = ( \; X(s_i),$

$\ldots, \; X(s_{in_i}))^T$, $ik = ( \; k(s_i), \ldots, \; k(s_{in_i}))^T$,

$ik$ $k$ $^T_{ik}\boldsymbol{\Sigma}^-_{U_i}(\mathbf{U}_i - \; X_i)$ $(j,l)$

$n_i \times n_i$ $\boldsymbol{\Sigma}_{U_i}$ $(\boldsymbol{\Sigma}_{U_i})_{j,l} = G_X(s_{ij}, s_{il}) + \; _X\delta_{jl}$ $\delta_{jl} =$

$j = l$ $\delta_{jl} =$ $j \neq l$. $ik$

$X_i$ $k$ $ik$ $\boldsymbol{\Sigma}_{X_i}$ (

)

$$\hat{}_{ik} = \hat{}_k \hat{}^T_{ik} \widehat{\boldsymbol{\Sigma}}^-_{U_i} (\mathbf{U}_i - \hat{}_{X_i}),$$

$(j,l)$ $\widehat{\boldsymbol{\Sigma}}_{U_i}$ $(\widehat{\boldsymbol{\Sigma}}_{U_i})_{j,l} = \widehat{G}_X(s_{ij}, s_{il}) +$

$\hat{}_Y(s_{ij})\delta_{jl}$. M ( )

( E ) $\hat{}_{ik}, \hat{\zeta}_{im}$

( )

$$\hat{}^I_{ik} = \sum_{j=}^{n_i} (U_{ij} - \hat{}_X(t_{ij})) \; \hat{}_k(s_{ij})(s_{ij} - s_{i,j-}),$$

$$\hat{\zeta}^I_{im} = \sum_{j=}^{n_i} (V_{ij} - \hat{}_Y(t_{ij})) \; \hat{}_k(t_{ij})(t_{ij} - t_{i,j-}),$$

$ik = \int \{X_i(s) - \; X(s)\} \; k(s)\,ds$ $\zeta_{im} = \int \{Y_i(t) -$

$Y(t)\}$ $m(t)\,dt$. E

A.2

( ) E (

)E $X$

$$\sum_{i=}^{n} \sum_{j=}^{n_i} K \left(\frac{s_{ij} - s}{b_X}\right) \{U_{ij} - \beta^X - \beta^X (s - s_{ij})\}$$

$\beta^X$ $\beta^X$ $\hat{}_X(s) = \hat{\beta}^X(s)$. $K$

$b_X$

E $G_X(s, s)$

$X$ $^X_i(s_{ij}, s_{ij}) = (U_{ij} - \hat{}_X(s_{ij}))(U_{ij} -$

$\hat{}_X(s_{ij}))$

$$\sum_{i=}^{n} \sum_{\leq j \neq j \leq n_i} K \left(\frac{s_{ij} - s}{h_X}, \frac{s_{ij} - s}{h_X}\right)$$

$$\times \left\{ \; ^X_i(s_{ij}, s_{ij}) - f\big(\beta^X, (s, s), (s_{ij}, s_{ij})\big)\right\},$$

$f(\beta^X, (s, s), (s_{ij}, s_{ij})) = \beta^X + \beta^X (s - s_{jl}) + \beta^X (s -$

$s_{ij})$ $^X = (\beta^X, \beta^X, \beta^X)^T$, $\widehat{G}_X(s, s) =$

$\hat{\beta}^X(s, s)$. H $K$

$h_X$

$j = j$

$X$

$G_X$

$\widetilde{G}_X(s)$

$\{G_X(s,s) + \; _X\}$ $\widehat{V}_X(s)$

$h^*_X$ $a = \{s\ s \in \mathcal{S}\}$ $b = \{s\ s \in \mathcal{S}\}$
$|\mathcal{S}| = b - a$ $\mathcal{S} = [a + |\mathcal{S}|/, b - |\mathcal{S}|/]$ $X$

$$\hat{}_X = \int \{\widehat{V}_X(s) - \widetilde{G}_X(s)\}\, ds/|\mathcal{S}| \qquad (\ )$$

$\hat{}_X >$ $\hat{}_X =$ E
$\{\ _k,\ _k\}_{k\geq}$ $\{\hat{}_k, \hat{}$

$k$ $t \in \mathcal{T}$

( ) $\sum_{k=}^{K} [|f_k''(\ _k)|h_k + n^{-\ /} \{\ (Y|\ _k)\}^{\ /} p_k^{-\ /}(\ _k) \times h_k^{-\ /}] \to$

( ) $\sum_{k=}^{K} \sum_{m=}^{M} [|f_{mk}''(\ _k)\ _m(t)|h_{mk} + n^{-\ /} \{\ (\zeta_m|\ _k)\}^{\ /} \times p_k^{-\ /}(\ _k)|\ _m(t)|h_{mk}^{-\ /}] \to$

*[Received August 2007. Revised June 2008.]*

M E E M E H K M M K ( ) E *Drosophila melanogaster*. *Science*

M ( ) *Topics in Stochastic Processes*

( ) *Probability and Measure* ( )

H ( ) *The Annals of Statistics*

H ( ) E M *Journal of Multivariate Analysis*

H M ( ) H M *Scandinavian Journal of Statistics*

M ( ) *Canadian Journal of Statistics*

# Supplement for "Functional Additive Models"

## 1. NOTATIONS AND AUXILIARY RESULTS

Covariance operators are denoted by $\mathcal{G}_X$, $\widehat{\mathcal{G}}_X$, generated by kernels $G_X$, $\widehat{G}_X$, i.e., $\mathcal{G}_X(f) = \int_{\mathcal{T}} G_X(s,t) f(s)ds$, $\widehat{\mathcal{G}}_X(f) = \int_{\mathcal{T}} \widehat{G}_X(s,t) f(s)ds$ for any $f \in L^2(\mathcal{T})$. Define

$$D_X = \int_{\mathcal{T}^2} \{\widehat{G}_X(s,t) - G_X(s,t)\}^2 dsdt, \qquad \delta_k^X = \min_{1\le j \le k}(\lambda_j - \lambda_{j+1}),$$

$$K_0 = \inf\{j \ge 1: \lambda_j - \lambda_{j+1} \le 2D_X\} - 1, \qquad \pi_k^X = 1/\lambda_k + 1/\delta_k^X. \quad (1)$$

Let $K = K(n)$ denote the number of leading eigenfunctions included to approximate $X$ [illegible] sample size $n$, i.e., $\widehat{X}_i(s) = \mu_X(s) + \sum_{k=1}^{K} \xi_{ik}\phi_k(s)$. Analogously, define the quantities $\mathcal{G}_Y$, $\widehat{\mathcal{G}}_Y$, $D_Y$, $\delta_m^Y$, $\pi_m^Y$, $M_0$ and $M$ for the process $Y$, for the case of functional responses. The following [illegible] convergence [illegible] for the estimators of the [illegible]. [illegible] the [illegible] developments in the proofs in Section [illegible].

**Lemma 1** *Under (A[illegible])–(A[illegible]), (C[illegible])–(C[illegible]) and (C[illegible]),*

$$\sup_{t\in\mathcal{S}} |\hat\mu_X(s) - \mu_X(s)| = O_p\left(\frac{1}{\sqrt{n}b_X}\right), \quad \sup_{s_1,s_2\in\mathcal{S}} |\widehat{G}_X(s_1,s_2) - G_X(s_1,s_2)| = O_p\left(\frac{1}{\sqrt{n}h_X^2}\right), \quad (2)$$

*and as a consequence* $\hat\sigma_X^2 - \sigma_X^2 = O_p(n^{-1/2}h_X^{-2} + n^{-1/2}h_X^{*-1})$. *Considering eigenvalues* $\lambda_k$ *of multiplicity one,* $\hat\phi_k$ *can be chosen such that*

$$P\left(\sup_{1\le k\le K_0} |\hat\lambda_k - \lambda_k| \le D_X\right) = 1, \quad \sup_{s\in\mathcal{S}} |\hat\phi_k(s) - \phi_k(s)| = O_p\left(\frac{\pi_k^X}{\sqrt{n}h_X^2}\right), \quad k = 1,\ldots,K_0, \quad (3)$$

*where* $D_X$, $\pi_k^X$ *and* $K_0$ *are defined in (1).*

*Analogously, under (B[illegible])–(B[illegible]), (C[illegible])–(C[illegible]) and (C[illegible]),*

$$\sup_{t\in\mathcal{T}} |\hat\mu_Y(t) - \mu_Y(t)| = O_p\left(\frac{1}{\sqrt{n}b_Y}\right), \quad \sup_{t_1,t_2\in\mathcal{T}} |\widehat{G}_Y(t_1,t_2) - G_Y(t_1,t_2)| = O_p\left(\frac{1}{\sqrt{n}h_Y^2}\right), \quad (4)$$

*and as a consequence* $\hat\sigma_Y^2 - \sigma_Y^2 = O_p(n^{-1/2}h_Y^{-2} + n^{-1/2}h_Y^{*-1})$. *Considering eigenvalues* $\rho_m$ *of multiplicity one,* $\hat\psi_m$ *can be chosen such that*

$$P\left(\sup_{1\le m\le M_0} |\hat\rho_m - \rho_m| \le D_Y\right) = 1, \quad \sup_{t\in\mathcal{T}} |\hat\psi_m(t) - \psi_m(t)| = O_p\left(\frac{\pi_k^Y}{\sqrt{n}h_Y^2}\right), \quad m = 1,\ldots,M_0, \quad (5)$$

*here* $D_{\mathbf{Y}}$ $\pi^{\mathbf{y}}_{\mathbf{k}}$ *nd* $M$ *re de ned n ogous y to for process* $Y$

e $\|f\|_\infty$ sup$_{\mathbf{x}\in\mathcal{A}}|f(t)|$ fo an y nc on $f$ ppo $\mathcal{A}$, and $\|g\|$ $\sqrt{\int_{\mathcal{A}} g^2(t)dt}$ fo any $g \in L^2(\mathcal{A})$ and de ne

$$
\begin{aligned}
&\theta^{(1)}_{\mathbf{ik}} \quad c_1\|X_{\mathbf{i}}\| \quad c_2\|X_{\mathbf{i}}X'_{\mathbf{i}}\|_\infty \quad {}^{*}_{\mathbf{X}} \quad c_3, && Z^{(1)}_{\mathbf{k}} \quad \sup_{\mathbf{s}\in\mathcal{S}}|\phi_{\mathbf{k}}(s) - \phi_{\mathbf{k}}(s)|,\\
&\theta^{(2)}_{\mathbf{ik}} \quad \|\phi_{\mathbf{k}}\phi'_{\mathbf{k}}\|_\infty \quad {}^{*}_{\mathbf{X}}, && Z^{(2)}_{\mathbf{k}} \quad \sup_{\mathbf{s}\in\mathcal{S}}|\mu_{\mathbf{X}}(s) - \mu_{\mathbf{X}}(s)|,\\
&\theta^{(3)}_{\mathbf{ik}} \quad c_4\|X_{\mathbf{i}}\|_\infty \quad c_5\|X'_{\mathbf{i}}\|_\infty \quad c_6, && Z^{(3)}_{\mathbf{k}} \quad \|\phi'_{\mathbf{k}}\|_\infty \quad {}^{*}_{\mathbf{X}},\\
&\theta^{(4)}_{\mathbf{ik}} \quad |\textstyle\sum_{\mathbf{j}=2}^{\mathbf{n}_i}\epsilon_{\mathbf{ij}}\phi_{\mathbf{k}}(s_{\mathbf{ij}})(s_{\mathbf{ij}} - s_{\mathbf{i,j}-1})|, && Z^{(4)}_{\mathbf{k}} \equiv,\\
&\theta^{(5)}_{\mathbf{ik}} \quad \textstyle\sum_{\mathbf{j}=2}^{\mathbf{n}_i}|\epsilon_{\mathbf{ij}}|(s_{\mathbf{ij}} - s_{\mathbf{i,j}-1}) \; i_{\mathbf{k}} && j
\end{aligned}
$$

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**Lemma 2** *For $\theta_{ik}^{()}$ $Z_{k}^{()}$ $\vartheta_{im}^{()}$ nd $Q_{m}^{()}$ s de ned in nd*

$$|\xi_{ik}^{I} - \xi_{ik}| \leq \sum_{=1}^{5} \theta_{ik}^{()} Z_{k}^{()}, \qquad |\zeta_{im}^{I} - \zeta_{im}| \leq \sum_{=1}^{5} \vartheta_{im}^{()} Q_{m}^{()}.$$

e p oof n ec on n e e e pp e e pe c p $I$ n e C e e $\xi_{ik}^{I}$ and $\zeta_{im}^{I}$

e e ence of nd d $h_{k}$ and $h_{mk}$ e e ployed o o n e e e $f_{k}$ and $f_{mk}$ fo e e e on nc on $f_{k}$ and $f_{mk}$, and e den y of $\xi_{k}$ deno ed y $p_{k}$ e ne

$$\theta_{k}(x) \quad p_{k}(x)\{\frac{\pi_{k}^{x}}{\sqrt{n}h_{X}^{2}} \quad \frac{}{\sqrt{n}b_{X}} \quad \sqrt{\quad {}_{X}^{*}}\},$$
$$\vartheta_{mk}(x) \quad p_{k}(x)\{\frac{\pi_{m}^{y}}{\sqrt{n}h_{Y}^{2}} \quad \frac{}{\sqrt{n}b_{Y}} \quad \sqrt{\quad {}_{Y}^{*}}\}.$$

e con e ence e $\theta_{k}$ and $\vartheta_{mk}$ of e e e on nc on e o $f_{k}(x)$ and $f_{mk}(x)$ ee eo e e fo o

$$\theta_{k}(x) \quad \frac{\theta_{k}(x)}{h_{k}} \quad -|f_{k}''(x)|h_{k}^{2} \quad \sqrt{\frac{\quad Y|(x)\|K_{1}\|^{2}}{p_{k}(x)nh_{k}}},$$
$$\vartheta_{mk}(x) \quad \frac{\theta_{k}(x)}{h_{mk}} \quad \vartheta_{mk}(x) \quad -|f_{mk}''(x)|h_{mk}^{2} \quad \sqrt{\frac{\quad \zeta_{m}|(x)\|K_{1}\|^{2}}{p_{k}(x)nh_{mk}}}.$$

Con de n e p ed c on $\widehat{E}(Y|X)$ fo e e pon e e and $\widehat{E}\{Y(t)|X\}$ fo e nc on e pon e e, e n e of e en nc on $K$ and $M$ ed fo app o n e n n e d en on p oce e $X$ and $Y$ ene y end o n n y e ple ze $n$ nc e e e e $K \leq K_0$ and $M \leq M_0$ n A() nce fo o f o () $K_0 \to \infty$, on e en e $\lambda_{j}$ e of p c y , and n o y fo $M_0$, no on e c on eno e e e of po e n e e y $\mathcal{N}$ and $\mathcal{N}_{k}$ $\{, \ldots, k\}$ Con e ence e $\theta_{n}^{*}$ and $\vartheta_{n}^{*}$ fo e p ed c on () and [illegible] e[illegible]s f[illegible]o[illegible]s

$$\theta_{\mathbf{n}}^{*} \quad \sum_{\mathbf{k}=1}^{\mathbf{K}}\Big\{\frac{\theta_{\mathbf{k}}\,\xi_{\mathbf{k}}}{h_{\mathbf{k}}} \quad -|f''_{\mathbf{k}}\,\xi_{\mathbf{k}}|h_{\mathbf{k}}^{2} \quad \sqrt{\frac{Y|\xi_{\mathbf{k}}|\|K_1\|^2}{p_{\mathbf{k}}\,\xi_{\mathbf{k}}\,n h_{\mathbf{k}}}}\Big\} \quad \Big|\sum_{\mathbf{k}\geq \mathbf{K}+1} f_{\mathbf{k}}\,\xi_{\mathbf{k}}\Big|,$$

$$\vartheta_{\mathbf{n}}^{*} \quad \sum_{\mathbf{k}=1}^{\mathbf{K}}\sum_{\mathbf{m}=1}^{\mathbf{M}}\Big\{\frac{\theta_{\mathbf{k}}\,\xi_{\mathbf{k}}}{h_{\mathbf{mk}}} \quad \vartheta_{\mathbf{mk}}\,\xi_{\mathbf{k}}|\psi_{\mathbf{m}}\,t| \quad -|f''_{\mathbf{mk}}\,\xi_{\mathbf{k}}|\psi_{\mathbf{m}}\,t|h_{\mathbf{mk}}^{2} \quad \sqrt{\frac{\zeta_{\mathbf{m}}|\xi_{\mathbf{k}}|\|K_1\|^2}{p_{\mathbf{k}}\,\xi_{\mathbf{k}}\,n h_{\mathbf{k}}}}|\psi_{\mathbf{m}}\,t|$$

$$\frac{\pi_{\mathbf{m}}^{\mathbf{y}}|f_{\mathbf{mk}}\,\xi_{\mathbf{k}}|}{\quad}$$

No n $\xi_{ik}$ $\eta_{ik}$ $\tau_{ik}$, one find s

$$|\xi_{ik} - \xi_{ik}| \leq \{|\eta_{ik} - \eta_{ik}| \quad |\eta_{ik} - \xi_{ik}| \quad |\tau_{ik}|\}.$$

o ss of ene y, ss e $\|\phi_k\|_\infty \geq$, $\|\phi'_k\|_\infty \geq$, $\|X_i\|_\infty \geq$ and $\|X'_i\|_\infty \geq$ o $\theta_{ik}^{(\ell)}$ and $Z_k^{(\ell)}$, $\ell$, …, , e s e on e s of s nded y

$$\{\sum_{j=2}^{n_i} |X_i s_{ij} - \mu s_{ij}| \cdot |\phi_k s_{ij} - \phi_k s_{ij}| \quad |\mu s_{ij} - \mu s_{ij}| \cdot |\phi_k s_{ij}| \; s_{ij} - s_{i,j-1}\}$$

$$\leq \{\sum_{j=1}^{n_i} |X_i s_{ij}| \quad |\mu s_{ij}| \quad {}^2 \; s_{ij} - s_{i,j-1}\}^{1/2} \{\sum_{j=2}^{n_i} \phi_k s_{ij} - \phi_k s_{ij} {}^2 \; s_{ij} - s_{i,j-1}\}^{1/2}$$

$$\{\sum_{j=1}^{n_i} \mu s_{ij} - \mu s_{ij} {}^2 \; s_{ij} - s_{i,j-1}\}^{1/2} \{\sum_{j=2}^{n_i} \phi_k^2 s_{ij} \; s_{ij} - s_{i,j-1}\}^{1/2}$$

$$\leq \theta_{ik}^{(1)} Z_k^{(1)} \quad \theta_{ik}^{(2)} Z_k^{(2)}.$$

e second e on e s of s e ppe nd

$$|\eta_{ij} - \xi_{ik}| \leq \| X_i \quad \mu' \phi_k \quad X_i \quad \mu \phi'_k \|_\infty \quad \overset{*}{X} \leq \theta_{ik}^{(3)} Z_k^{(3)}.$$

o e e, e d e on e s of s nded y $\theta_{ik}^{(4)} Z_k^{(4)} \quad \theta_{ik}^{(5)} Z_k^{(5)}$

□

*Proof of Theorem* o s p c y, deno e $\sum_{i=1}^{n}$ y $\sum_i$, $w_i$ $K_1\{x - \xi_{ik}/h_k\}/nh_k$, $w_i$ $K_1\{x - \xi_{ik}/h_k\}/nh_k$, and e $\theta_k$ $\theta_k x$ o , e oc ne e s o $f_k x$ of e e e ss on nc on $f_k x$ an e e p c y en s

$$f_k \quad \frac{\sum_i w_i Y_i}{\sum_i w_i} - \frac{\sum_i w_i \xi_{ik} - x}{\sum_i w_i} f'_k x,$$

e e

$$f'_k x \quad \frac{\sum_i w_i \xi_{ik} - x Y_i - \{\sum_i w_i \xi_{ik} - x \sum_i w_i Y_i\}/\sum_i w_i}{\sum_i w_i \xi_{ik} - x^2 - \{\sum_i w_i \xi_{ik} - x\}^2/\sum_i w_i}.$$

Le $f_k x$ e ypo e e s o , o ned y s s n e e s $w_i$ and $\xi_{ik}$ fo $w_i$, $\xi_{ik}$ n and o e e $|f_k x - f_k x|$, one s o n fy e o de s of the d e ence s

$$D_1 \quad \sum_i (w_i - \hat{w}_i), \quad D_2 \quad \sum_i (w_i - \hat{w}_i) Y_i,$$

$$D_3 \quad \sum_i (w_i \xi_{ik} - w_i \hat{\xi}_{ik}), \quad D_4 \quad \sum_i (w_i \xi^2_{ik} - w_i \hat{\xi}^2_{ik}).$$

Con s de n $D_1$, the o o ss of ene a y, a ss e the co pac s ppo of $K_1$ s , nce $K_1$ s L p s z con n o s on s s ppo ,

$$D_1 \leq \frac{c}{nh^2_k} \sum_i |\hat{\xi}_{ik} - \xi_{ik}| \{ I(|x - \xi_{ik}| \leq h_k) \quad I(|x - \hat{\xi}_{ik}| \leq h_k) \},$$

fo so e $c >$ , the e $I(\cdot)$ s an nd ca o nc on Le a p e s fo the s e on the s of

$$\frac{1}{nh^2_k} \sum_i |\hat{\xi}_{ik} - \xi_{ik}| I(|x - \xi_{ik}| \leq h_k) \leq \sum_{\ell=1}^{5} Z^{(\ell)}_k \frac{1}{nh^2_k} \sum_i \theta^{(\ell)}_{ik} I(|x - \xi_{ik}| \leq h_k).$$

Apply n the cen a the eo e fo a ando n e of s and s B n ey, 1995, p a e , o s n $\sum_i I(|x - \xi_{ik}| \leq h_k)/(nh_k) \xrightarrow{p} p_k(x)$, one nd s

$$\frac{1}{nh_k} \sum_i \theta^{(\ell)}_{ik} I(|x - \xi_{ik}| \leq h_k) \xrightarrow{p} p_k(x) E(\theta^{(\ell)}_{ik}),$$

p o ded that $E(\theta^{(\ell)}_{ik}) < \infty$ fo $\ell$ 1, . . . , No e that $E\theta^{(1)}_{ik} < \infty$, $E\theta^{(3)}_{ik} < \infty$ y A , $E\theta^{(4)}_{ik} \leq \sigma_X \sqrt{\ \ ^*_X}$ and $E\theta^{(5)}_{ik} \leq |\mathcal{S}| \sigma_X$ y the Ca chy ch a z ne a y then

$$Z^{(1)}_k \frac{1}{nh^2_k} \sum_i \theta^{(1)}_{ik} I(|x - \xi_{ik}| \leq h_k) \quad O_p\{ \frac{\pi^X_k}{\sqrt{n} h^2_X h_k} p_k(x) \},$$

$$Z^{(2)}_k \frac{1}{nh^2_k} \sum_i \theta^{(2)}_{ik} I(|x - \xi_{ik}| \leq h_k) \quad O_p\{ \frac{1}{\sqrt{n} b_X h_k} p_k(x) \},$$

$$Z^{(3)}_k \frac{1}{nh^2_k} \sum_i \theta^{(3)}_{ik} I(|x - \xi_{ik}| \leq h_k) \quad O_p\{ \frac{\|\phi_k\|_\infty \ \ ^*_X}{h_k} p_k(x) \},$$

$$Z^{(4)}_k \frac{1}{nh^2_k} \sum_i \theta^{(4)}_{ik} I(|x - \xi_{ik}| \leq h_k) \quad O_p\{ \frac{\sqrt{\ \ ^*_X}}{h_k} p_k(x) \},$$

$$Z^{(5)}_k \frac{1}{nh^2_k} \sum_i \theta^{(5)}_{ik} I(|x - \xi_{ik}| \leq h_k) \quad O_p\{ \frac{\pi^X_k}{\sqrt{n} h^2_X h_k} p_k(x) \}.$$

We now show that $(nh_k)^{-1}\sum_i |\hat\xi_{ik}-\xi_{ik}| I(|x-\xi_{ik}|\le h_k) = O_p(\theta_k h_k)$. The key proof of the second term can be derived analogously. Observing

$$\frac{1}{nh_k}\sum_i I(|x-\hat\xi_{ik}|\le h_k) \le \frac{1}{nh_k}\sum_i \{I(|x-\xi_{ik}|\le 2h_k) + I(\sum_{\ell=1}^{5}\theta_{ik}^{(\ell)} Z_k^{(\ell)} > h_k)\} \xrightarrow{p} 2p_k(x),$$

and noting $(nh_k)^{-1}\sum_i |\hat\xi_{ik}-\xi_{ik}| I(|x-\xi_{ik}|\le h_k) = O_p(\theta_k h_k)$, then $D_1 = O_p(\theta_k h_k)$ follows.

Analogously, one shows $D_2 = O_p(\theta_k h_k)$, applying the Cauchy–Schwarz inequality for $\theta_{ik}^{(\ell)}$, $\ell = 1, 2$, and observing the independence between $Y_i$ and $\theta_{ik}^{(\ell)}$ for $\ell = 3, 4, 5$, given the moment condition (A). For $D_3$, observe

$$D_3 = \sum_i \{(w_i - \hat w_i)\xi_{ik} + (w_i - \hat w_i)(\hat\xi_{ik} - \xi_{ik}) + w_i(\hat\xi_{ik} - \xi_{ik})\} \equiv D_{31} + D_{32} + D_{33}.$$

Then $D_{31} = O_p(\theta_k h_k)$, analogously to $D_1$. It is easy to see that $D_{32} = o_p(D_{31})$. Since $D_{33} \le c\sum_{\ell=1}^{5} Z_k^{(\ell)} (nh_k)^{-1}\sum_i \theta_{ik}^{(\ell)} I(|x-\xi_{ik}|\le h_k)$ for some $c>0$, one also has $D_{33} = o_p(D_{31})$. This results in $D_3 = O_p(\theta_k h_k)$. Observing $|\hat\xi_{ik}^2 - \xi_{ik}^2| \le |\hat\xi_{ik}-\xi_{ik}| \cdot |\xi_{ik}| + (\hat\xi_{ik}-\xi_{ik})^2$, one can show $D_4 = O_p(\theta_k h_k)$, using similar arguments as for $D_3$, and $E\xi_{ik}^4 < \infty$ from (A). Combining the results for $D_\ell$, $\ell = 1, \dots, 4$, and applying Slutsky's theorem lead to $|\hat f_k(x) - f_k(x)| = O_p(\theta_k h_k)$ in (A), and applying standard asymptotic results for the hypothesis in the local linear smoother $f_k(x)$ completes the proof of (1).

To derive (9), additionally one only needs to consider $\sum_i \hat w_i\hat\zeta_{im} - w_i\zeta_{im} = \sum_i \{(w_i - \hat w_i)\zeta_{im} + (w_i - \hat w_i)(\hat\zeta_{im} - \zeta_{im}) + w_i(\hat\zeta_{im} - \zeta_{im})\}$, where the leading term is dominated by the third term of order $O_p(\vartheta_{mk})$ by observing

$$|\sum_i w_i(\hat\zeta_{im} - \zeta_{im})| \le \sum_{\ell=1}^{5} Q_m^{(\ell)} \sum_i w_i\vartheta_{im}^{(\ell)} \le \frac{1}{nh_{mk}}\sum_{\ell=1}^{5} Q_m^{(\ell)} \sum_i \vartheta_{im}^{(\ell)} I(|x-\xi_{ik}|\le h_{mk}).$$

Similar arguments then lead to complete the derivation. □

*Proof of Theorem 2.* Using (A), the derivation of $\theta_n^*$ in (4) is standard, follow in the proof of Theorem 3 and note that

$$
\begin{aligned}
&\widehat{E}\{Y(t)|X\} - E\{Y(t)|X\} \\
\le\ & \sum_{\mathbf{k}=1}^{\mathbf{K}} \sum_{\mathbf{m}=1}^{\mathbf{M}} |f_{\mathbf{mk}}(\xi_{\mathbf{k}})\psi_{\mathbf{m}}(t) - f_{\mathbf{mk}}(\xi_{\mathbf{k}})\psi_{\mathbf{m}}(t)| + |\sum_{\mathbf{k}\ge \mathbf{K}+1} \sum_{\mathbf{m}\ge \mathbf{M}+1} f_{\mathbf{mk}}(\xi_{\mathbf{k}})\psi_{\mathbf{m}}(t)| \\
\le\ & \sum_{\mathbf{k}=1}^{\mathbf{K}} \sum_{\mathbf{m}=1}^{\mathbf{M}} |f_{\mathbf{mk}}(\xi_{\mathbf{k}}) - f_{\mathbf{mk}}(\xi_{\mathbf{k}})|\{|\psi_{\mathbf{m}}(t)| + |\psi_{\mathbf{m}}(t) - \psi_{\mathbf{m}}(t)|\} + |f_{\mathbf{mk}}(\xi_{\mathbf{k}})| \cdot |\psi_{\mathbf{m}}(t) - \psi_{\mathbf{m}}(t)| \\
& + |\sum_{(\mathbf{k},\mathbf{m})\in\mathcal{N}^2\setminus\mathcal{N}_K\times\mathcal{N}_M} f_{\mathbf{mk}}(\xi_{\mathbf{k}})\psi_{\mathbf{m}}(t)|.
\end{aligned}
$$

This implies the convergence of $\vartheta_{\mathbf{n}}^*$ in (...).