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A mp o ic di ib ion of nonpa ame ic eg e ion e ima o fo longi dinal o f nc ional da a

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Abstract

The e ima ion of a eg e ion f nc ion b ke nel me hod fo longi dinal o f nc ional da a i con ide ed. In he con e of longi dinal da a anal i , a andom f nc ion picall ep e en a bjec ha i of en ob e ed a a mall n mbe of ime poin w hile in he die of f nc ional da a he andom eali a ion i all mea ed on a den e g id. Ho w e e , e en iall he ame me hod can be applied o bo h ampling plan , a w ell a in a n mbe of e ing l ing be w een hem. In hi pape gene al e l a e de i ed fo he a mp o ic di ib ion of eal- al ed f nc ion w i h a g men w hich a e f nc ional fo med b w eigh ed a e age of longi dinal o f nc ional da a. A mp o ic di ib ion fo he e ima o of he mean and co a iance f nc ion ob ained f om noi ob e a ion w i h he p e ence of w i hin- bjec co ela ion a e died. The e a mp o ic no mali e l a e compa able o ho e anda d a e ob ained f om independen da a w hich i ill a ed in a im la ion d . Be ide , hi pape di c e he condi ion a ocia ed w i h ampling plan w hich a e eq i ed fo he alidi of local p ope ie of ke nel-ba ed e ima o fo longi dinal o f nc ional da a.

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Keywords: A mp o ic di ib ion; Co a iance; F nc ional da a; Longi dinal da a; Reg e ion; Wi hin- bjec co ela ion

1. Introduction

Mode n echnolog and ad anced comp ing en i onmen ha e facili a ed he collec ion and anal i of high-dimen ional da a, o da a ha a e epea edl mea ed fo a ample of bjec . The epea ed mea emen a e of en eco ded o e a pe iod of ime, a on an clo ed and bo nded in e al $\mathcal{T}$. I al o co ld be a pacial a iable, ch a in image o geo cience applica ion .

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When he da a a e eco ded den el o e ime, of en b machine, he a e picall e med f nc-ional o c e da a w i h one ob e ed c e o f nc ion pe bjec w hile in longi dinal die he epea ed mea emen all ake place on a fe w ca e ed ob e a ional ime poin fo each bjec . A ignifican in in ic diffe ence be w een w o e ing lie in he pe cep ion ha f nc ional da a a e ob e ed in he con in m w i ho noi e [2,3] w he ea longi dinal da a a e ob e ed a pa el di ib ed ime poin and a e of en bjec o e pe imen al e o [4]. Ho w e e , in p ac-ice f nc ional da a a e anal ed af e moo hing noi ob e a ion [10] w hich indica e ha he diffe ence be w een w o da a pe ela ed o he w a in w hich a p oblem i pe cei ed a e a g abl mo e concep al han ac al. The efo e in hi pape , ke nel-ba ed eg e ion e ima o ob ained f om ob e a ion a di c e e ime poin con amina ed w i h mea emen e o , a he han ob e a ion in he con in m, a e con ide ed fo he e eali ic ea on . In he con e of ke nel-ba ed nonpa ame ic eg e ion, he effec of ampling plan on he a i ical e ima o a e al o in e iga ed.

A a li e a e ha been de eloped in he pa decade on he ke nel-ba ed eg e ion fo independen and iden icall di ib ed da a, fo mma , ee Fan and Gijbel [5]. The e ha been b an ial ecen in e e in e ending he e i ing a mp o ic e l o f nc ional o lon-gi dinal da a [8,11,14,13,9]. The i e ca ed b he w i hin- bjec co ela ion a e igo o l add e ed in hi pape . Ha and Weh l [8] died he Ga e Melle e ima o of he mean f nc ion fo epea ed mea emen ob e ed on a eg la g id b a ming a iona co ela-ion c e, and ho w ed ha he infl ence of he w i hin- bjec co ela ion on he a mp o ic a iance i of malle o de compa ed o he anda d a e ob ained f om independen da a and w ill di appea w hen he co ela ion f nc ion i diffe en iable a e o. O a mp o ic di ib ion e l i in fac con i en w i h ha in Ha and Weh l [8] and applicable fo gene al co a i-ance c e w i ho a iona a mp ion. Thi p oblem w a al o di c ed b S ani w ali and Lee [12] and Lin and Ca oll [9] w he e he ed he he i ic a g men of he local p ope of local pol nomial e ima ion and in i i el igno ed he w i hin- bjec co ela ion w hile de i ing he a mp o ic a iance . Thi pape de i e app op ia e condi ion ha a e e-q i ed fo he alidi of he local p ope of ke nel pe e ima o ob ained f om longi di-nal o f nc ional da a. The e condi ion al o p o ide p ac ical g ideline fo a io ampling p oced e .

The con ib ion of hi pape i he de i a ion of gene al a mp o ic di ib ion e l in bo h one-dimen ional and w o-dimen ional moo hing con e fo eal- al ed f nc ion w i h a -g men w hich a e f nc ional fo med b w eigh ed a e age of longi dinal o f nc ional da a. The e a mp o ic no mali e l a e compa able o ho e ob ained fo iden icall di ib ed and independen da a. The e e l a e applied o he ke nel-ba ed e ima o of he mean and co a iance f nc ion w hich ield a mp o ic no mal di ib ion of he e e ima o . In pa ic-la , o he be of o kno w ledge, no a mp o ic di ib ion e l a e a ailable p o da e fo nonpa ame ic e ima ion of co a iance f nc ion ob ained f om longi dinal o f nc ional da a con amina ed w i h mea emen e o . B compa i on, Hall e al. [6,7] in e iga ed a mp o ic p ope ie of nonpa ame ic ke nel e ima o of a oco a iance w he e he mea emen w e e onl ob e ed f om a ingle a iona ocha ic p oce o andom field. Al ho gh he a mp-o ic di ib ion a e de i ed fo andom de ign in hi pape , he a g men can be e ended o fi ed de ign and o he ampling plan w i h app op ia e modifica ion , and a mp o ic bia and a iance e m can al o be ob ained in imila manne . Thi w ill p o ide heo e ical ba i and p ac ical g idance fo he nonpa ame ic anal i of f nc ional o longi dinal da a w i h impo an po en ial applica ion w hich a e ba ed on he a mp o ic di ib ion . T pical e ample incl de he con c ion of a mp o ic confidence band fo eg e ion f nc ion and confidence egion fo co a iance face , and al o fa elec ion of band id h fo co a iance face e ima ion ba ed on a mp o ic mean q a ed e o . O he applica ion in he con e of moo hing indepen-den da a can be e plo ed fo he moo hing of longi dinal o f nc ional da a ing ke nel-ba ed e ima o .

The emainde of he pape i o gani ed a follo . In Sec ion 2, e de i e he gene al a mp o ic di ib ion of one- and o-dimen ional moo he ob ained f om longi dinal o f nc ional da a fo andom de ign. The e gene al a mp o ic e l a e applied o commonl ed ke nel- pe e ima o of he mean c e and co a iance face in Sec ion 3. E en ion o fi ed de ign i di c ed in Sec ion 4. A im la ion d i p e en ed o e al a e he de i ed a mp o ic e l fo co ela ed da a in Sec ion 5, hile di c ion , incl ding po en ial applica ion of he e l ing a mp o ic no mali , a e offe ed in Sec ion 6.

2. General results of asymptotic distributions for random design

In hi ec ion, e ill define gene al f nc ional ha a e ke nel- eigh ed a e age of he da a fo one-dimen ional and o-dimen ional moo hing. The in od ced gene al f nc ional incl de he mo commonl ed pe of ke nel-ba ed e ima o a pecial ca e , ch a Ga e Melle e ima o , Nada a a Wa on e ima o , local pol nomial e ima o , e c. Since Nada a a Wa on and local pol nomial e ima o a e mo l ed in p ac ice, hei a mp o ic beha io in e m of bia and a iance fo independen da a ha e been ho o ghl died in e i ing li e a e. Ho e e , fo longi dinal o f nc ional da a, pa ic la l in e-ga d o co a iance face e ima o , he a mp o ic beha io of bia and a iance of he e o commonl ed e ima o a e ill la gel nkno n. The efo e in Sec ion 3, he gen-e al a mp o ic e l de eloped in hi ec ion a e applied o Nada a a Wa on and local pol nomial e ima o in bo h one-dimen ional and o-dimen ional moo hing e ing . In pa ic la , he lack of a mp o ic e l fo he co a iance face e ima o of longi di-nal o f nc ional da a i an addi ional mo i a ion fo he defini ion of he o-dimen ional gene al f nc ional ha can be applied o de elop he a mp o ic di ib ion fo he e e ima o .

We fi con ide andom de ign hile e en ion o o he ampling plan i defe ed o Sec ion 4. In cla ical longi dinal die , mea emen a e of en in ended o be on a eg la ime g id. Ho e e , ince indi id al ma mi ched led i i , he e l ing da a all become pa e, he e onl fe ob e a ion a e ob ained fo mo bjec i h neq al n mbe of epea ed mea emen pe bjec and diffe en mea emen ime T_{ij} pe indi id al. Thi ampling variance σ^2,

$$Y_{ij} = X_i(T_{ij}) + \varepsilon_{ij} = \mu(T_{ij}) + \sum_{k=1}^{\infty} \xi_{ik}\phi_k(T_{ij}) + \varepsilon_{ij}, \quad T_{ij} \in \mathcal{T}, \tag{1}$$

where $E\varepsilon_{ij} = 0$, $var(\varepsilon_{ij}) = \sigma^2$, and the numbers of observations $N_i(n)$ depending on the sample size n, are considered random. We make the following assumptions,

(A1.1) The number of observations $N_i(n)$ made for the ith subject or cluster, $i = 1, \dots, n$, is a r.v. with $N_i(n) \overset{\text{i.i.d}}{\sim} N(n)$, where $N(n) > 0$ is a positive integer-valued random variable with $\limsup_{n\to\infty} EN(n)^2/[EN(n)]^2 < \infty$ and $\limsup_{n\to\infty} EN(n)^4/EN(n)EN(n)^3 < \infty$.

In the sequel the dependence of $N_i(n)$ and $N(n)$ on the sample size n is suppressed for implicity; i.e., $N_i = N_i(n)$ and $N(n) = N$. The observation times and measurements are assumed to be independent of the number of measurements, i.e., for any subset $J_i \subseteq \{1, \dots, N_i\}$ and for all $i = 1, \dots, n$,

(A1.2) $(\{T_{ij} : j \in J_i\}, \{Y_{ij} : j \in J_i\})$ is independent of N_i.

Writing $\boldsymbol{T}_i = (T_{i1}, \dots, T_{iN_i})^T$ and $\boldsymbol{Y}_i = (Y_{i1}, \dots, Y_{iN_i})^T$, it is easy to see that the triples $\{\boldsymbol{T}_i, \boldsymbol{Y}_i, N_i\}$ are i.i.d..

2.1. Asymptotic normality of one-dimensional smoother

To assume appropriate regularity conditions that are used to derive asymptotic properties, we define a new type of continuity that differs from those which are commonly used. We say that a real function $f(x, y) : \Re^{p+q} \to \Re$ is continuous on $x \in A \subseteq \Re^p$ uniformly in $y \in \Re^q$, provided that for any $x \in A$ and $\varepsilon > 0$, there exists a neighborhood of x not depending on y, saying $U(x) \subseteq \Re^p$, such that $|f(x', y) - f(x, y)| < \varepsilon$ for all $x' \in U(x)$ and $y \in \Re^q$.

For random design, (T_{ij}, Y_{ij}) are assumed to have the identical distribution as (T, Y) with joint density $g(t, y)$. Assume that the observation times T_{ij} are i.i.d. with the marginal density $f(t)$, but dependence is allowed among Y_{ij} and Y_{ik} that are observations made for the same subject or cluster. Also denote the joint density of (T_j, T_k, Y_j, Y_k) by $g_2(t_1, t_2, y_1, y_2)$ where $j \neq k$. Let ν, k be given integers with $0 \leqslant \nu < k$. We assume regularity conditions for the marginal and joint densities, $f(t)$, $g(t, y)$, $g_2(t_1, t_2, y_1, y_2)$ and the mean function of the underlying process $X(t)$, i.e., $E[X(t)] = \mu(t)$ with respect to a neighborhood of an interior point $t \in \mathcal{T}$, assuming that there exists a neighborhood $U(t)$ of t such that:

(B1.1) $\frac{d^k}{du^k} f(u)$ exists and is continuous on $u \in U(t)$, and $f(u) > 0$ for $u \in U(t)$;

(B1.2) $g(u, y)$ is continuous on $u \in U(t)$ uniformly in $y \in \Re$; $\frac{d^k}{du^k} g(u, y)$ exists and is continuous on $u \in U(t)$ uniformly in $y \in \Re$;

(B1.3) $g_2(u, v, y_1, y_2)$ is continuous on $(u, v) \in U(t)^2$ uniformly in $(y_1, y_2) \in \Re^2$;

(B1.4) $\frac{d^k}{du^k} \mu(u)$ exists and is continuous on $u \in U(t)$.

Let $K_1(\cdot)$ be nonnegative univariate kernel function in one-dimensional smoothing. The assumptions for kernel $K_1 : \Re \to \Re$ are as follows. We say that a univariate kernel function K_1 is of order (ν, k), if

$$\int u^\ell K_1(u)\, du = \begin{cases} 0, & 0 \leqslant \ell < k, \quad \ell \neq \nu, \\ (-1)^\nu \nu!, & \ell = \nu, \\ \neq 0, & \ell = k, \end{cases} \tag{2}$$

(B2.1) K_1 is compact supported, $\|K_1\|^2 = \int K_1^2(u)\,du < \infty$;
(B2.2) K_1 is a kernel function of order (v, ℓ).

Let $b = b(n)$ be a sequence of bandwidths that are used in one-dimensional smoothing. We develop asymptotics as $n \to \infty$, and require

(B3) $b \to 0$, $n(EN)b^{v+1} \to \infty$, $b(EN) \to 0$, and $n(EN)b^{2k+1} \to d^2$ for some d with $0 \leqslant d < \infty$.

One could see in the proof of Theorem 1 that the assumption (B3) combined with (A1.1) provide the conditions such that the local properties of kernel-type estimators hold for longitudinal or functional data with the presence of within-subject correlation.

Let $\{\psi_\lambda\}_{\lambda=1,\ldots,l}$ be a collection of real functions $\psi_\lambda : \Re^2 \to \Re$, which satisfy:

(B4.1) $\psi_\lambda(t, y)$ are continuous on $\{t\}$ uniformly in $y \in \Re$;
(B4.2) $\frac{d^k}{dt^k}\psi_\lambda(t, y)$ exist for all arguments (t, y) and are continuous on $\{t\}$ uniformly in $y \in \Re$.

Then we define the general weighted averages

$$\Psi_{\lambda n} = \frac{1}{nENb^{v+1}} \sum_{i=1}^{n} \sum_{j=1}^{N_i} \psi_\lambda(T_{ij}, Y_{ij}) K_1\left(\frac{t - T_{ij}}{b}\right), \quad \lambda = 1, \ldots, l.$$

and

$$\mu_\lambda = \mu_\lambda(t) = \frac{d^v}{dt^v} \int \psi_\lambda(t, y) g(t, y)\,dy, \quad \lambda = 1, \ldots, l.$$

Let

$$\sigma_{\kappa\lambda} = \sigma_{\kappa\lambda}(t) = \int \psi_\kappa(t, y)\psi_\lambda(t, y) g(t, y)\,dy \|K_1\|^2, \quad 1 \leqslant \lambda, \kappa \leqslant l,$$

and $H : \Re^l \to \Re$ be a function with continuous first order derivatives. We denote the gradient vector $((\partial H/\partial x_1)(v), \ldots, (\partial H/\partial x_l)(v))^T$ by $DH(v)$ and $\bar{N} = \sum_{i=1}^n N_i/n$.

Theorem 1. *If the assumptions* (A1.1), (A1.2) *and* (B1.1)–(B4.2) *hold*, *then*

$$\sqrt{n\bar{N}b^{2v+1}}[H(\Psi_{1n}, \ldots, \Psi_{ln}) - H(\mu_1, \ldots, \mu_l)] \xrightarrow{\mathcal{D}} \mathcal{N}(\beta, [DH(\mu_1, \ldots, \mu_l)]^T \Sigma[DH(\mu_1, \ldots, \mu_l)]), \tag{3}$$

where

$$\beta = \frac{(-1)^k d}{k!} \int u^k K_1(u)\,du \sum_{\lambda=1}^{l} \frac{\partial H}{\partial \mu_\lambda}\{(\mu_1, \ldots, \mu_l)^T\} \frac{d^{k-v}}{dt^{k-v}} \mu_\lambda(t), \quad \Sigma = (\sigma_{\kappa\lambda})_{1 \leqslant \kappa, \lambda \leqslant l}.$$

Proof. It is seen that $\bar{N}$ can be replaced with EN by Slutsky Theorem under (A1.1). We now show that

$$\sqrt{n(EN)b^{2v+1}}[H(E\Psi_{1n}, \ldots, E\Psi_{ln}) - H(\mu_1, \ldots, \mu_l)] \longrightarrow \beta. \tag{4}$$

Since (A1.1) and (A1.2) hold, and K_1 i of o de (v, k), ing Ta lo e pan ion o o de k, one ob ain

$$E\Psi_{\lambda n} = \frac{1}{nb^{v+1}} E\left\{\sum_{i=1}^{n} \frac{1}{EN} \sum_{j=1}^{N_i} \psi_\lambda(T_{ij}, Y_{ij}) K_1\left(\frac{t-T_{ij}}{b}\right)\right\}$$

$$= \frac{1}{b^{v+1}EN} E\left\{\sum_{j=1}^{N} E\left[\psi_\lambda(T_j, Y_j) K_1\left(\frac{t-T_j}{b}\right)\middle| N\right]\right\}$$

$$= \frac{1}{b^{v+1}} E\left\{\psi_\lambda(T, Y) K_1\left(\frac{t-T}{b}\right)\right\}$$

$$= \mu_\lambda + \frac{(-1)^k}{k!} \int u^k K_1(u)\, du\, \frac{d^{k-v}}{dt^{k-v}} \mu_\lambda(t) b^{k-v} + o(b^{k-v}). \tag{5}$$

Then (4) follo f om an l-dimen ional Ta lo e pan ion of H of o de 1 a o nd $(\mu_1, \ldots, \mu_l)^T$, co pled i h (5). If e can ho

$$\sqrt{n(EN)b^{2v+1}}[(\Psi_{1n}, \ldots, \Psi_{ln})^T - (E\Psi, \ldots, E\Psi_{ln})^T] \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma), \tag{6}$$

in analog o Bha acha a and Melle [1], and con in i of DH a $(\mu_1, \ldots, \mu_l)^T$ and appl - ing imila a g men ed in (5), e find $DH(E\Psi_{1n}, \ldots, E\Psi_{ln}) \to DH(\mu_1, \ldots, \mu_l)$. Then Ca mè Wold de ice ield

$$\sqrt{n(EN)b^{2v+1}}[H(\Psi_{1n}, \ldots, \Psi_{ln}) - H(E\Psi, \ldots, E\Psi_{ln})] \xrightarrow{\mathcal{D}} \mathcal{N}(0, DH(\mu_1, \ldots, \mu_l)^T \Sigma DH(\mu_1, \ldots, \mu_l)), \tag{7}$$

combined i h (4), leading o (3).

I emain o ho (6). Ob e ing (A1.1) and (A1.2), one ha

$$\begin{aligned} &n(EN)b^{2v+1} cov(\Psi_{\lambda n}, \Psi_{\kappa n}) \\ &\quad = \frac{1}{b} E\left\{\frac{1}{EN}\left[\sum_{j=1}^{N} \psi_\lambda(T_j, Y_j) K_1\left(\frac{t-T_j}{b}\right)\right]\left[\sum_{k=1}^{N} \psi_\kappa(T_k, Y_k) K_1\left(\frac{t-T_k}{b}\right)\right]\right\} \\ &\qquad - \frac{EN}{b} E\left[\frac{1}{EN} \sum_{j=1}^{N} \psi_\lambda(T_j, Y_j) K_1\left(\frac{t-T_j}{b}\right)\right] \\ &\qquad \times E\left[\frac{1}{EN} \sum_{k=1}^{N} \psi_\kappa(T_k, Y_k) K_1\left(\frac{t-T_k}{b}\right)\right] \\ &\quad \equiv I_1 - I_2. \end{aligned}$$

I i ob io ha $I_2 = O(b) = o(1)$ f om he de i a ion of (5). Fo I_1, i can bew i en a

$$I_1 = \frac{1}{b} E\left[\frac{1}{EN}\sum_{j=1}^{N} \psi_\lambda(T_j, Y_j)\psi_\kappa(T_j, Y_j) K_1^2\left(\frac{t-T_j}{b}\right)\right]$$
$$+\frac{1}{b} E\left[\frac{1}{EN}\sum_{1\leqslant j\neq k\leqslant N} \psi_\lambda(T_j, Y_j)\psi_\kappa(T_k, Y_k) K_1\left(\frac{t-T_j}{b}\right) K_1\left(\frac{t-Y_k}{b}\right)\right]$$
$$\equiv Q_1 + Q_2.$$

Appl ing (A1.1) and (A1.2), one ha

$$Q_1 = \frac{1}{b} E\left\{\frac{1}{EN}\sum_{j=1}^{N} E\left[\psi_\lambda(T_j, Y_j)\psi_\kappa(T_j, Y_j) K_1^2\left(\frac{t-T_j}{b}\right)\Big| N\right]\right\}$$
$$= \frac{1}{b} E\left[\psi_\lambda(T, Y)\psi_\kappa(T, Y) K_1^2\left(\frac{t-Y}{b}\right)\right] = \sigma_{\lambda\kappa} + o(1).$$

Then (4) w ill hold, ob e ing (A1.1) and he follow ing a g men ha g a an ee he local p ope of he ke nel-ba ed e ima o w i h he p e ence of w i hin- bjec co ela ion in longi dinal o f nc ional da a,

$$Q_2 = \frac{1}{bEN} E\left\{\sum_{1\leqslant j\neq k\leqslant N}^{N} E\left[\psi_\lambda(T_j, Y_j)\psi_\kappa(T_k, Y_k) K_1\left(\frac{t-T_j}{b}\right) K_1\left(\frac{t-T_k}{b}\right)\Big| N\right]\right\}$$
$$= \frac{EN(N-1)}{bEN} E\left[\psi_\lambda(T_1, Y_1)\psi_\kappa(T_2, Y_2) K_1\left(\frac{t-T_1}{b}\right)\right] K_1\left(\frac{t-T_2}{b}\right)$$
$$= \frac{bEN(N-1)}{EN}\int_{\Re^4} \psi_\lambda(t-ub, y_1)\psi_\kappa(t-vb, y_2) K_1(u) K_2(v)$$
$$\times g_2(t-ub, t-vb, y_1, y_2)\, du\, dv\, dy_1\, dy_2$$
$$= \frac{bEN(N-1)}{EN}\int_{\Re^2} \psi_\lambda(t, y_1)\psi_\kappa(t, y_2) g_2(t, t, y_1, y_2)\, dy_1\, dy_2 + o(b) = o(1),$$

i.e., hew i hin- bjec co ela ion can be igno edw hile de i ing he a mp o ic a iance. □

2.2. Asymptotic normality of two-dimensional smoother

The gene al a mp o ic e l can be e ended o w o-dimen ional moo hing. Le $(\boldsymbol{v}, \boldsymbol{k})$ deno e he m l i-indice $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{k} = (k_1, k_2)$, w he e $|\boldsymbol{v}| = v_1 + v_2$ and $|\boldsymbol{k}| = k_1 + k_2$. In w o-dimen ional moo hing, mo e eg la i a mp ion a e needed fo join den i ie . Le $f_2(s, t)$ be he join den i of (T_j, T_k), and $g_4(s, t, s', t', y_1, y_2, y_1', y_2')$ he join den i of $(T_j, T_k, T_{j'}, T_{k'}, Y_j, Y_k, Y_{j'}, Y_{k'})$, w he e $j \neq k$, $(j, k) \neq (j', k')$. Deno e he co a iance -face b $C(s, t) = cov(X(T_j), X(T_k)|T_j = s, T_k = t)$. The follow ing eg la i condi ion a e a medw he e $U(s, t)$ i ome neighbo hood of $\{(s, t)\}$,

(C1.1) $\frac{d^{|\boldsymbol{k}|}}{du^{k_1}\, dv^{k_2}} f_2(u, v)$ e i and i con in o on $(u, v) \in U(s, t)$, and $f_2(u, v) > 0$ fo $(u, v) \in U(s, t)$;

(C1.2) $g_2(u, v, y_1, y_2)$ is continuous on $(u, v) \in U(s, t)$ uniformly in $(y_1, y_2) \in \Re^2$; $\frac{d^{|\boldsymbol{k}|}}{du^{k_1}\, dv^{k_2}} g_2(u, v, y_1, y_2)$ exists and is continuous on $(u, v) \in U(s, t)$ uniformly in $(y_1, y_2) \in \Re^2$;

(C1.3) $g_4(u, v, u', v', y_1, y_2, y_1', y_2')$ is continuous on $(u, v, u', v') \in U(s, t)^2$ uniformly in $(y_1, y_2, y_1', y_2') \in \Re^4$;

(C1.4) $\frac{d^{|\boldsymbol{k}|}}{du^{k_1}\, dv^{k_2}} C(u, v)$ exists and is continuous on $(u, v) \in U(s, t)$.

Let K_2 be nonnegative bivariate kernel function used in the two-dimensional smoothing. The assumptions for kernel K_2 are as follows,

(C2.1) K_2 is compact supported with $\|K_2\|^2 = \int_{\Re^2} K_2^2(u, v)\, du\, dv < \infty$, and is symmetric with respect to coordinate u and v.

(C2.2) K_2 is a kernel function of order $(|\boldsymbol{v}|, |\boldsymbol{k}|)$, i.e.,

$$\sum_{\ell_1+\ell_2=|\boldsymbol{l}|} \int_{\Re^2} u^{\ell_1} v^{\ell_2} K_2(u, v)\, du\, dv = \begin{cases} 0, & 0 \leqslant |\boldsymbol{l}| < |\boldsymbol{k}|, |\boldsymbol{l}| \neq |\boldsymbol{v}|, \\ (-1)^{|\boldsymbol{v}|}|\boldsymbol{v}|!, & |\boldsymbol{l}| = |\boldsymbol{v}|, \\ \neq 0, & |\boldsymbol{l}| = |\boldsymbol{k}|. \end{cases} \tag{8}$$

Let $h = h(n)$ be a sequence of bandwidth used in two-dimensional smoothing, while it is possible that the bandwidth used for two arguments may be different. Since we will focus on the estimator of the covariance surface that is symmetric about the diagonal, it is sufficient to consider the identical bandwidth for the two arguments. The asymptotic is developed as $n \to \infty$ as follows:

(C3) $h \to 0$, $nEN^2 h^{|\boldsymbol{v}|+2} \to \infty$, $hEN^3 \to 0$, and $nE[N(N-1)]h^{2|\boldsymbol{k}|+2} \to e^2$ for some $0 \leqslant e < \infty$.

Similar to the one-dimensional smoothing case, assumption (C3) and (A1.1) guarantee the local property of the bivariate kernel-based estimator with the presence of within-subject correlation.

Let $\{\phi_\lambda\}_{\lambda=1,\ldots,l}$ be a collection of real functions $\phi_\lambda : \Re^4 \to \Re$, $\lambda = 1, \ldots, l$, satisfying

(C4.1) $\phi_\lambda(s, t, y_1, y_2)$ are continuous on $\{(s, t)\}$ uniformly in $(y_1, y_2) \in \Re^2$;

(C4.2) $\frac{d^{|\boldsymbol{k}|}}{ds^{k_1} dt^{k_2}} \phi_\lambda(s, t, y_1, y_2)$ exist for all arguments (s, t, y_1, y_2) and are continuous on $\{(s, t)\}$ uniformly in $(y_1, y_2) \in \Re^2$.

Then the general weighted averages of two-dimensional smoothing are defined by, for $1 \leqslant \lambda \leqslant l$,

$$\Phi_{\lambda n} = \Phi_{\lambda n}(t, s) = \frac{1}{nE[N(N-1)]h^{|\boldsymbol{v}|+2}} \sum_{i=1}^{n} \sum_{1 \leqslant j \neq k \leqslant N_i} \phi_\lambda(T_{ij}, T_{ik}, Y_{ij}, Y_{ik})$$

$$\times K_2\left(\frac{s - T_{ij}}{h}, \frac{t - T_{ik}}{h}\right).$$

Let

$$m_\lambda = m_\lambda(s, t) = \sum_{v_1+v_2=|\boldsymbol{v}|} \frac{d^{|\boldsymbol{v}|}}{ds^{v_1}\, dt^{v_2}} \int_{\Re^2} \phi_\lambda(s, t, y_1, y_2) g_2(s, t, y_1, y_2)\, dy_1\, dy_2, \quad 1 \leqslant \lambda \leqslant l,$$

and

$$\omega_{\kappa\lambda} = \omega_{\kappa\lambda}(s,t) = \int_{\Re^2} \phi_\kappa(s,t,y_1,y_2)\phi_\lambda(s,t,y_1,y_2)g_2(s,t,y_1,y_2)dy_1dy_2\|K_2\|^2,$$

$$1 \leqslant \kappa, \lambda \leqslant l,$$

and $H : \Re^l \to \Re$ i a f nc ion w i h con in o fi o de de i a i e a p e io 1 defined.

Theorem 2. *If assumptions* (A1.1), (A1.2) *and* (C1.1) (C4.2) *hold*, *then*

$$\sqrt{n\bar{N}(\bar{N}-1)h^{2|\boldsymbol{\nu}|+2}}[H(\Phi_{1n},\ldots,\Phi_{ln}) - H(m_1,\ldots,m_l)]$$

$$\xrightarrow{\mathcal{D}} \mathcal{N}(\gamma, [DH(m_1,\ldots,m_l)]^T\Omega[DH(m_1,\ldots,m_l)]), \tag{9}$$

where

$$\gamma = \frac{(-1)^{|\boldsymbol{k}|}e}{|\boldsymbol{k}|!}\sum_{\lambda=1}^{l}\left\{\sum_{k_1+k_2=|\boldsymbol{k}|}\int_{\Re^2} u^{k_1}v^{k_2}K_2(u,v)\,du\,dv\frac{d^{|\boldsymbol{k}|}}{ds^{k_1}\,dt^{k_2}}\right.$$

$$\left.\times\int_{\Re^2}\phi_\lambda(s,t,y_1,y_2)g_2(s,t,y_1,y_2)\,dy_1\,dy_2\right\}$$

$$\times\left\{\frac{\partial H}{\partial m_\lambda}(m_1,\ldots,m_l)^T\right\},$$

$$\Omega = (\omega_{\kappa\lambda})_{1\leqslant\kappa\leqslant l}.$$

The p oof of Theo em 2 e en iall follo w ha of Theo em 1 w i h app op ia e modifica ion w hich a e eq i ed fo w o-dimen ional moo hing.

3. Applications to nonparametric regression estimators for functional or longitudinal data

Al ho gh a io e ion of ke nel-ba ed e ima o ha e been in od ced in li e a e, Nada a a Wa on and local pol nomial, e peciall local linea e ima o , a e he mo commonl ed non-pa ame ic moo hing echniq e in longi dinal o f nc ional da a anal i . D e o w i hin- bjec co ela ion, he a mp o ic beha io in e m of bia and a iance of he e e ima o fo noi il ob e ed longi dinal o f nc ional da a ha e been a w ell nde ood a fo i.i.d. da a. E peciall , a mp o ic e l fo co a iance e ima o do no e i . The efo e in hi ec ion w e appl he a mp o ic e l de eloped fo gene al f nc ional o Nada a a Wa on and local linea e ima o of eg e ion f nc ion and co a iance face o ob ain hei a mp o ic di ib ion .

3.1. Asymptotic distributions of mean estimators

We appl Theo em 1 o he local a mp o ic di ib ion of he commonl ed Nada a a Wa on ke nel e ima o $\hat{\mu}_{\mathrm{N}}(t)$ and local linea e ima o $\hat{\mu}_{\mathrm{L}}(t)$ fo f nc ional/longi dinal da a:

$$\hat{\mu}_{\mathrm{N}}(t) = \left[\sum_{i=1}^{n}\sum_{j=1}^{N_i} K_1\left(\frac{t-T_{ij}}{b}\right) Y_{ij}\right] \Bigg/ \left[\sum_{i=1}^{n}\sum_{j=1}^{N_i} K_1\left(\frac{t-T_{ij}}{b}\right)\right], \tag{10}$$

$$\hat{\mu}_{\mathrm{L}}(t) = \hat{\alpha}_0(t) = \mathrm{a\ g\,min}_{(\alpha_0,\,\alpha_1)} \left\{\sum_{i=1}^{n}\sum_{j=1}^{N_i} K_1\left(\frac{t-T_{ij}}{b}\right)[Y_{ij} - (\alpha_0 + \alpha_1(T_{ij}-t))]^2\right\}. \tag{11}$$

Corollary 1. *If assumptions* (A1.1), (A1.2), *and* (B1.1) (B3) *hold with* $v = 0$ *and* $k = 2$, *then*

$$\sqrt{n\bar{N}b}[\hat{\mu}_{\mathrm{N}}(t) - \mu(t)] \xrightarrow{\mathcal{D}} \mathcal{N}\left(\frac{d}{2}\frac{\mu^{(2)}(t)f(t) + 2\mu^{(1)}(t)f^{(1)}(t)}{f(t)}\sigma_{K_1}^2, \frac{var(Y|T=t)\|K_1\|^2}{f(t)}\right), \tag{12}$$

where d is as in (B3), $\sigma_{K_1}^2 = \int u^2 K_1(u)\,du$

Here $w_{ij} = K_1((t-T_{ij})/b)/(nb)$, where K_1 is a kernel function of order $(0,2)$, satisfying (B2.1) and (B2.2), and $\hat{\alpha}_1(t)$ is an estimator for the first derivative $\mu'(t)$ of μ at t.

Observing that Corollary 1 implies $\hat{\mu}_{\mathrm{N}}(t) \xrightarrow{p} \mu(t)$, let $\hat{f}(t) = \sum_i \sum_j w_{ij}/N_i$, it is easy to show $\hat{f}(t) \xrightarrow{p} f(t)$ in analogy to Corollary 1. We proceed to show $\hat{a}_1(t) \xrightarrow{p} \mu'(t)$. Denote $\sigma^2_{K_1} = \int u^2 K_1(u)\, du$, the kernel function $\widetilde{K}_1(t) = -tK_1(t)/\sigma^2_{K_1}$, and define $\Psi_{\lambda n}$, $1 \leqslant \lambda \leqslant 3$ by $\psi_1(u,y) = y$, $\psi_2(u,y) \equiv 1$, $\psi_3(u,y) = u - t$. Observe that $\widetilde{K}_1$ is of order $(1,3)$, $\hat{f}(t) \xrightarrow{p} f(t)$, and define

$$\widetilde{H}(x_1,x_2,x_3) = \frac{x_1 - x_2\hat{\mu}_{\mathrm{N}}(t)}{x_3 - bx_2^2/\hat{f}(t)\cdot\sigma^2_{K_1}} \quad \text{and} \quad H(x_1,x_2,x_3) = \frac{x_1 - x_2\mu(t)}{x_3}.$$

Then

$$\begin{aligned}\hat{\alpha}_1(t) &= \widetilde{H}(\Psi_{1n}, \Psi_{2n}, \Psi_{3n}) \\ &= \left[H(\Psi_{1n},\Psi_{2n},\Psi_{3n}) + \frac{\Psi_{2n}(\mu(t) - \hat{\mu}_{\mathrm{N}}(t))}{\Psi_{3n}}\right] \frac{\Psi_{3n}}{\Psi_{3n} + b^2\Psi_{2n}^2/\hat{f}(t)\cdot\sigma^2_{K_1}}.\end{aligned}$$

Note that $\mu_1 = (\mu' f + m f')(t)$, $\mu_2 = f'(t)$, and $\mu_3 = f(t)$, implying $\Psi_{\lambda n} - \mu_\lambda = O_p(1/\sqrt{n\bar{N}b^3})$, for $\lambda = 1,2,3$, by Theorem 1. Using *Slutsky's* Theorem, $|\widetilde{H}(\Psi_{1n},\Psi_{2n},\Psi_{3n}) - \mu'(t)| = O_p(1/\sqrt{n\bar{N}b^3})$ follows.

For the asymptotic distribution of $\hat{\mu}_{\mathrm{L}}$, note that

$$\hat{\mu}_{\mathrm{L}}(t) = \frac{\sum_i \frac{1}{EN}\sum_j w_{ij}Y_{ij} - \sum_i \frac{1}{EN}\sum_j w_{ij}(T_{ij}-t)\hat{a}_1(t)}{\sum_i \frac{1}{EN}\sum_j w_{ij}}.$$

Considering $\sqrt{n\bar{N}b}\sum_i \frac{1}{EN}\sum_j w_{ij}(T_{ij}-t) = \sqrt{n\bar{N}b}\sigma^2_{K_1}b^2\Psi_{2n}$. Since $\tilde{K}_1$ is of order $(1,3)$, Theorem 1 implies $\Psi_{2n} = f'(t) + O_p(1/\sqrt{n\bar{N}b^3})$, which yields $\sqrt{n\bar{N}b}\sigma^2_{K_1}b^2\Psi_{2n} = \sqrt{n\bar{N}b^5}\sigma^2_{K_1} f'(t) + \sigma^2_{K_1}O_p(b) = o_p(1)$ by observing $n\bar{N}b^5 \to d^2$ for $0 \leqslant d < \infty$. Since $\hat{f}(t) \xrightarrow{p} f(t)$ and $|\hat{\alpha}_1(t) - \mu'(t)| = O_p(1/\sqrt{n\bar{N}b^3}) = o_p(1)$, we find

$$\begin{aligned}\lim_{n\to\infty} \sqrt{n\bar{N}b}[\hat{\mu}_{\mathrm{L}}(t) - \mu(t)] &\overset{\mathcal{D}}{=} \lim_{n\to\infty} \sqrt{n\bar{N}b} \\ &\times \left\{ \frac{\sum_i \frac{1}{EN}\sum_j w_{ij}Y_{ij} - \mu'(t)\sum_i \frac{1}{EN}\sum_j w_{ij}T_{ij} + t\mu'(t)\sum_i \frac{1}{EN}\sum_j w_{ij}}{\sum_i \frac{1}{EN}\sum_j w_{ij}} - \mu(t)\right\}.\end{aligned}$$

Using the kernel K_1 of order $(0,2)$, we re-define $\Psi_{\lambda n}$, $1 \leqslant \lambda \leqslant 3$, through $\psi_1(u,y) = y$, $\psi_2(u,y) = u$ and $\psi_3(u,y) \equiv 1$, setting $v = 0$, $k = 2$, $l = 3$ and $H(x_1,x_2,x_3) = [x_1 - \mu'(t)x_2 + t\mu'(t)x_3]/x_3$. Then (13) follows by applying Theorem 1. □

3.2. Asymptotic distributions of covariance estimators

Note that in model (1), $cov(Y_{ij}, Y_{ik}|T_{ij}, T_{ik}) = cov(X(T_{ij}), X(T_{ik})) + \sigma^2\delta_{jk}$, where δ_{jl} is 1 if $j = k$ and 0 otherwise. Let $C_{ijk} = (Y_{ij} - \hat{\mu}(T_{ij}))(Y_{ik} - \hat{\mu}(T_{ik}))$ be the raw covariances, where $\hat{\mu}(t)$ is the estimated mean function obtained from the previous step, for instance, $\hat{\mu}(t) = \hat{\mu}_{\mathrm{N}}(t)$ or $\hat{\mu}(t) = \hat{\mu}_{\mathrm{L}}(t)$. It is easy to see that $E[C_{ijk}|T_{ij}, T_{ik}] \approx cov(X(T_{ij}), X(T_{ik})) + \sigma^2\delta_{jk}$. Therefore, he diagonal of he a co a iance ho ld be emo ed, i.e., onl C_{ijk}, $j \neq k$, ho ld be incl ded a inp da a fo he co a iance face moo hing ep, a p e io l ob e ed in S ani ali and Lee [12] and Yao e al. [15].

Commonl ed nonpa ame ic eg e ion e ima o of he co a iance face, $C(s,t) = E\{[X(T_1)-\mu(T_1)][X(T_2)-\mu(T_2)|T_1 = s, T_2 = t]\}$, a e he o-dimen ional Nada a a Wa on e ima o and local linea e ima o defined a follo :

$$\widehat{C}_{\mathrm{N}}(s,t) = \left[\sum_{i=1}^{n}\sum_{j\neq k} K_2\left(\frac{s-T_{ij}}{h}, \frac{t-T_{ik}}{h}\right) C_{ijk}\right] \Bigg/ \left[\sum_{i=1}^{n}\sum_{j\neq k} K_2\left(\frac{s-T_{ij}}{h}, \frac{t-T_{ik}}{h}\right)\right], \tag{16}$$

$$\widehat{C}_{\mathrm{L}}(s,t) = \hat{\beta}_0(s,t) = \mathrm{a\ g\,min}_{\boldsymbol{\beta}} \left\{\sum_{i=1}^{n}\sum_{j\neq k} K_2\left(\frac{s-T_{ij}}{h}, \frac{t-T_{ik}}{h}\right) \times [C_{ijk} - f(\boldsymbol{\beta}, (s,t), (T_{ij}, T_{ik}))]\right.$$

$\phi_1(t_1,t_2,y_1,y_2) = (y_1-\mu(t_1))(y_2-\mu(t_2))$, $\phi_2(t_1,t_2,y_1,y_2) = y_1-\mu(t_1)$, and $\phi_3(t_1,t_2,y_1,y_2) \equiv 1$, hen p$_{t,s\in\mathcal{T}}$ $|\Phi_{pn}| = O_p(1)$, fo $p = 1,2,3$, b Lemma 1 of Yao e al. [16]. Thi implie ha p$_{t,s\in\mathcal{T}}$ $|\Phi_{2n}|O_p(1/(\sqrt{n}b)) = O_p(1/(\sqrt{n}b))$ and p$_{t,s\in\mathcal{T}}$ $|\Phi_{3n}|O_p(1/(\sqrt{n}b)) = O_p(1/(\sqrt{n}b))$. Since p$_{t\in\mathcal{T}}$ $|\hat{\mu}(t)-\mu(t)|^2 = O_p(1/(nb))$ a e negligible compa ed o Φ_{1n}, he Nada a a Wa on e ima o $\widehat{C}_{\mathrm{N}}(s,t)$, of $C(s,t)$ ob ained f om C_{ijk} i a mp o icall eq i alen o ha ob ained f om $\tilde{C}_{ijk}$, deno ed b $\tilde{C}_{\mathrm{N}}(t,s)$.

The efo e, i i fficien o ho ha he a mp o ic di ib ion of $\tilde{C}_{\mathrm{N}}(s,t)$ follo (18). Choo e $\boldsymbol{v} = (0,0)$, $|\boldsymbol{k}| = 2$, $\phi_1(s,t,y_1,y_2) = (y_1-\mu(s))(y_2-\mu(t))$, $\phi_2(s,t,y_1,y_2) \equiv 1$ and $H(x_1,x_2) = x_1/x_2$ in Theo em 2, hen $\tilde{C}_{\mathrm{N}}(s,t) = H(\Psi_{1n},\Psi_{2n})$. To comp e $\gamma_{\mathrm{N}}(s,t)$, e $DH(m_1,m_2) = (1/m_2, -m_1/m_2^2)$, and no e $m_1(s,t) = \int_{\Re^2}(y_1-\mu(s))(y_2-\mu(t))g_2(s,t,y_1,y_2)\,dy_1dy_2 = f_2(s,t)C(s,t)$ and $m_2(s,t) = f_2(s,t)$. One ha $(d^2/dt^2)m_1(s,t) = [(d^2f_2/dt^2)C + 2(df_2/dt)(dC/dt) + f_2(d^2C/dt^2)](s,t)$, $(d^2/d^2t)m_2(s,t) = d^2f_2(s,t)/dt^2$ and imila de i a i e i h e pec o he a g men s leading o he bia e m in (12). Fo he a mp o ic a i-ance, no e ha $\omega_{11} = \|K_2\|^2\int_{\Re^2}(y_1-\mu(s))^2(y_2-\mu(t))^2g_2(s,t,y_1,y_2)dy_1dy_2 = E[(Y_1-\mu(T_1))^2(Y_2-\mu(T_2))^2|T_1 = s, T_2 = t)f_2(s,t)\|K_2\|^2$, $\omega_{12} = \omega_{21} = \|K_2\|^2f_2(s,t)C(s,t)$, $\omega_{22} = \|K_2\|^2f_2(s,t)$, and $DH(m_1,m_2) = (1/m_2, -m_1/m_2^2)$, ielding he a iance e m in (12). □

Corollary 4. *If the assumptions* (A1.1), (A1.2), *and* (C1.1) (C3) *hold with* $|\boldsymbol{v}| = 0$ *and* $|\boldsymbol{k}| = 2$, *then*

$$\sqrt{n\bar{N}(\bar{N}-1)h^2}[\widehat{C}_{\mathrm{L}}(s,t) - C(s,t)]$$

$$\xrightarrow{\mathcal{D}} \mathcal{N}\left(\frac{e}{4}\sigma_{K_2}^2[d^2C(s,t)/ds^2 + d^2C(s,t)/dt^2], \frac{v(s,t)\|K_2\|^2}{f_2(s,t)}\right), \tag{19}$$

where e is as in (C3), $v(s,t) = var\{(Y_1-\mu(T_1))(Y_2-\mu(T_2))|T_1 = s, T_2 = t)$, $\sigma_{K_2}^2 = \int_{\Re^2}(u^2+v^2)K_2(u,v)\,du\,dv$, $\|K_2\|^2 = \int_{\mathcal{R}^2}K_2^2(u,v)\,du\,dv$.

Proof. In analog o he p oof of Co olla 3, he local linea e ima o $\widehat{C}_{\mathrm{L}}(s,t)$ ob ained f om C_{ijk} i a mp o icall eq i alen o ha ob ained f om $\tilde{C}_{ijk}$, deno ed b $\tilde{C}_{\mathrm{L}}(t,s)$. Al o deno e he ol ion o (17), af e b i ing $\tilde{C}_{ijk}$ fo C_{ijk}, b $\tilde{\boldsymbol{\beta}}(s,t) = (\tilde{\beta}_0(s,t), \tilde{\beta}_1(s,t), \tilde{\beta}_2(s,t))$, and in fac $\tilde{\beta}_0(s,t) = \tilde{C}_{\mathrm{L}}(s,t)$. Fo implici , le $W_{ijk} = K_2((s-T_{ij})/h, (t-T_{ik})/h)/(nh^2)$ and "$\sum_{i,j\neq k}$" i abb e ia ion of "$\sum_{i=1}^n\sum_{j\neq k}$". Algeb a calc la ion ield ha

$$\tilde{C}_{\mathrm{L}} = \frac{\sum_{i,j\neq k}\tilde{C}_{ijk}W_{ijk} - \tilde{\beta}_1\sum_{i,j\neq k}W_{ijk}T_{ij} + \tilde{\beta}_1\sum_{i,j\neq k}W_{ijk}s - \tilde{\beta}_2\sum_{i,j\neq k}W_{ijk}T_{ik} + \tilde{\beta}_2\sum_{i,j\neq k}W_{ijk}t}{\sum_{i,j\neq k}W_{ijk}},$$

$$\tilde{\beta}_1 = \frac{R_{00}(S_{10}S_{02}-S_{01}S_{11}) + R_{10}(S_{00}S_{02}-S_{01}S_{20}) - R_{01}(S_{00}S_{11}-S_{10}S_{02})}{S_{00}S_{20}S_{02} - S_{00}S_{11}^2 - S_{10}^2S_{02} + S_{10}S_{01}S_{11} + S_{20}S_{10}S_{11} - S_{01}S_{20}^2},$$

$$\tilde{\beta}_2 = \frac{R_{00}(S_{10}S_{11}-S_{01}S_{02}) - R_{10}(S_{00}S_{11}-S_{01}S_{20}) + R_{01}(S_{00}S_{20}-S_{10}^2)}{S_{00}S_{20}S_{02} - S_{00}S_{11}^2 - S_{10}^2S_{02} + S_{10}S_{01}S_{11} + S_{20}S_{10}S_{11} - S_{01}S_{20}^2},$$

he e

$$R_{pq} = \sum_{i,j\neq k}W_{ijk}(T_{ij}-s)^p(T_{ik}-t)^q\tilde{C}_{ijk}, \quad S_{pq} = \sum_{i,j\neq k}W_{ijk}(T_{ij}-s)^p(T_{ik}-t)^q.$$

No e ha $\tilde{\beta}_1$ and $\tilde{\beta}_2$ a e local linea e ima o of he pa ial de i a i e of $C(s,t)$, $dC(s,t)/ds$ and $dC(s,t)/dt$, e pec i el . In analog o he p oof of Co olla 2, i can be ho n ha $|\tilde{\beta}_1(s,t) - dC(s,t)/ds| = O_p(1/\sqrt{nEN(N-1)h^4})$ and $|\tilde{\beta}_2(s,t) - dC(s,t)/dt| = O_p(1/\sqrt{n\bar{N}(\bar{N}-1)h^4})$ b appl ing Theo em 2. Then one can b i e $dc(s,t)/ds$, $dC(s,t)/dt$ fo $\tilde{\beta}_1(s,t)$, $\tilde{\beta}_2(s,t)$ in $\tilde{C}_{\mathrm{L}}(s,t)$, and deno e he e l ing e ima o b $C^*_{\mathrm{L}}(s,t)$. I i ea o ee ha

$$\lim_{n\to\infty} \sqrt{n\bar{N}(\bar{N}-1)h^2}[C_{\mathrm{L}}(s,t) - C(s,t)] \stackrel{\mathcal{D}}{=} \lim_{n\to\infty} \sqrt{n\bar{N}(\bar{N}-1)h^2}[C^*_{\mathrm{L}}(s,t) - C(s,t)].$$

We define $\Phi_{\lambda n}$, $1 \leqslant \lambda \leqslant 4$, h o gh $\phi_1(s,t,y_1,y_2) = (y_1 - \mu(s))(y_2 - \mu(t))$, $\phi_2(s,t,y_1,y_2$., .))

ho e in Co olla ie 3 and 4w i h $f(t)$ eplaced b $1/|\mathcal{T}|$ and $f(s,t)$ eplaced b $1/|\mathcal{T}|^2$w he e $|\mathcal{T}|$ i he leng h of he in e al.

5. Simulation study

A n me ical d i cond c ed o e al a e he de i ed a mp o ic p ope ie . The ke finding in hi pape i ha he a mp o ic e l fo f nc ional o longi dinal a e compa able o ho e ob ained f om independen da a, i.e., he infl ence ofw i hin- bjec co a iance doe no pla ignifican ole in de e mining he a mp o ic bia and a iance. Fo implici w e foc on he local pol nomial mean e ima o w hich a e of en pe io o he Nada a a Wa on e ima o .

We fi gene a ed $M = 200$ ample con i ing of $n = 50$ i.i.d. andom ajec o ie each. Follow ing model (1), he im la ed p oce ha a mean f nc ion $\mu(t) = (t - 1/2)^2$, $0 \leqslant t \leqslant 1$ w hich ha a con an econd de i a i e $\mu^{(2)}(t) = 2$, and a con anw i hin- bjec co a iance f nc ion de i ed f om a andom in e cep $\xi_1 \overset{\text{i.i.d.}}{\sim} N(0, \lambda_1)$w he e $\lambda_1 = 0.01$ and$\phi_1(t) = 1$, $0 \leqslant t \leqslant 1$. The mea emen e o in (1)w a e $\varepsilon_{ij} \overset{\text{i.i.d.}}{\sim} N(0, \sigma^2)$w he e $\sigma^2 = 0.01$. A andom de ignw a edw he e he n mbe of ob e a ion fo each bjec N_iw e e cho en f om $\{2, 3, 4, 5\}$w i h eq al likelihood and he loca ion of he ob e a ionw e e nifo ml di ib ed on $[0, 1]$, i.e., $T_{ij} \overset{\text{i.i.d.}}{\sim} U[0, 1]$. Fo compa i onw e gene a ed $M = 200$ ample of $n = 50$ i.i.d. andom ajec o iew hich ha e he ame c e a in model (1) b now i hin- bjec co ela ion. Le ing $\xi_{i1} = 0$ and $\varepsilon_{ij} \overset{\text{i.i.d.}}{\sim} N(0, \sqrt{\lambda_1 + \sigma^2})$ lead o independen da aw i h he ame mean and a iance f nc ion . The efo e, hew o e of da a ha e he ame a mp o ic di ib ion fo he local pol nomial mean e ima o . We al o gene a ed $M = 200$ co ela ed and independen ample , e pec i el , con i ing of $n = 200$ ajec o ie each fo demon a ing he a mp o ic beha iow i h he inc ea ing ample i e n.

He ew e e he Epanechniko ke nel f nc ion, i.e., $K_1(u) = 3/4(1 - u^2)\mathbf{1}_{[-1,1]}(u)$w he e $\mathbf{1}_A(u) = 1$ if $u \in A$ and 0 o hew i e fo an e A. No e ha $n(EN)b^{2k+1} \to d^2$ in (B3), $\mu^{(2)}(t) = 2$, $var(Y|T = t) = \lambda_1 + \sigma^2 = 0.02$, and he de ign den i $f(t) = 1$w he e $k = 2$ fo local pol nomial e ima o and b i he bandw id h ed fo he mean e ima ion. F om he abo e con c ion, one can calc la e he a mp o ic a iance and bia of he local pol nomial mean e ima o $\hat{\mu}_{\mathrm{L}}(t)$ ing Co olla 2w hich i in fac applicable fo bo h co ela ed and independen da a. Since he bia and a iance e m a e bo h con an in o im la ion f amew o k, fo con eniencew e compa e he a mp o ic in eg a ed q a ed bia and a iance w i h he empi ical in eg a ed q a ed bia and a iance ob ained ing Mon e Ca lo a e age f om $M = 200$ im la ed ample ba ed on $\int_0^1 E[\{\hat{\mu}_{\mathrm{L}}(t) - \mu(t)\}^2]\,dt = \int_0^1 \{\hat{\mu}_{\mathrm{L}}(t) - E[\hat{\mu}_{\mathrm{L}}(t)]\}^2\,dt + \int_0^1 \{E[\hat{\mu}_{\mathrm{L}}(t)] - \mu(t)\}^2\,dt$. The a mp o ic in eg a ed q a ed bia and a iance a e gi en b

$$\text{AIBIAS} = \frac{1}{2}\sigma_{K_1}^2 b^4, \quad \text{AIVAR} = \frac{0.02 \times \|K_1\|^2}{n\bar{N}b}, \tag{20}$$

and he a mp o ic in eg a ed mean q a ed e o AIMSE = AIBIAS + AIVARw he e $\sigma_{K_1}^2 = \int u^2 K_1(u)\,du$, $\|K_1\|^2 = \int K_1^2(u)\,du$ and $\bar{N} = (1/n)\sum_{i=1}^n N_i$w hile he empi ical in eg a ed q a ed bia , a iance and mean q a ed e o a e deno ed b EIBIAS, EIVAR and EIMSE,

The a mp o ic and empi ical q an i ie , ch a he in eg a ed q a ed bia , a iance and mean q a ed e o , a e how n in Fig. 1 fo he co ela ed/independen da aw i h ample i e $n = 50/n = 200$, e pec i el . F om Fig. 1, i i ob io ha he a mp o ic app o ima ion i imp o ed b inc ea ing he ample i e. The a mp o ic q an i ie AIBIAS, AIVAR and AIMSE ag eew i h he

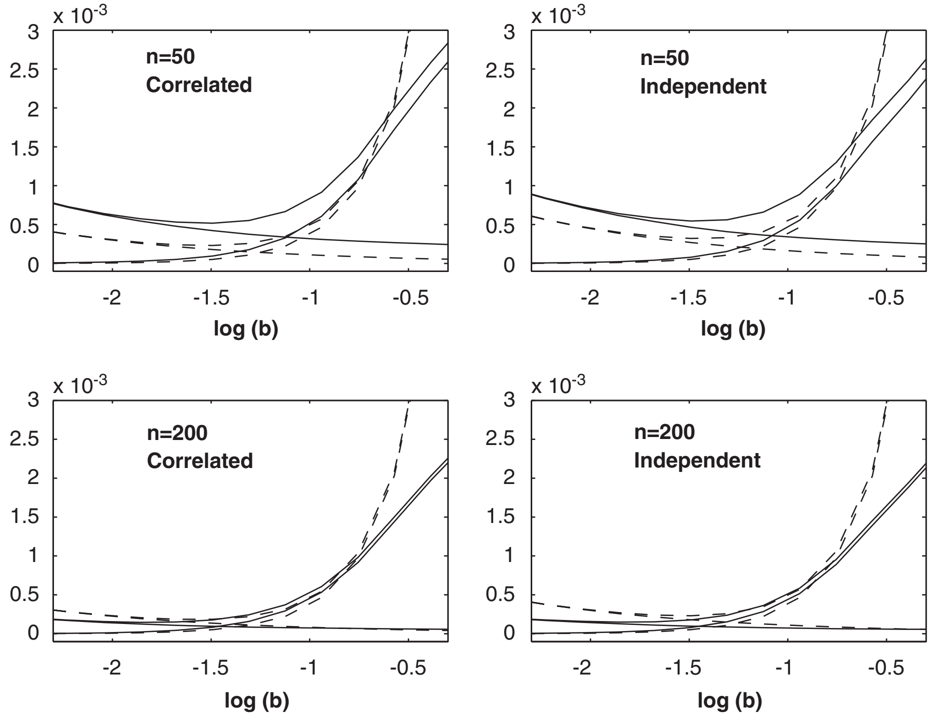

Fig. 1. Sho n a e he empi ical q an i ie (olid, incl ding EIBIAS, EIVAR, EIMSE) and a mp o ic q an i ie (da hed, incl ding AIBIAS, AIVAR, AIMSE) e log(b) fo co ela ed (lef panel) and independen (igh panel) da a i h diffe en ample i e $n = 50$ (op panel) and $n = 200$ (bo om panel) he e b i he band id h ed in he moo hing. In each panel, he in eg a ed q a ed bia i he one i h inc ea ing pa e n, he in eg a ed a iance i he one i h dec ea ing pa e n, and he c o each o he hile he in eg a ed mean q a ed e o hich i la ge han bo h in eg a ed q a ed bia and a iance fo an band id h b, all dec ea e fi and hen inc ea e af e eaching a minim m.

empi ical q an i ie EIBIAS, EIVAR and EIMSE fo bo h co ela ed and independen da a. Fo he im la ed da a i h he ame ample i e n, ch a mp o ic app o ima ion fo co ela ed and independen da a a e ell compa able in pa e n and magni de. Thi p o ide he e idence ha he i hin- bjec co ela ion indeed doe no ha e ob io infl ence on he a mp o ic beha io of he local pol nomial e ima o compa ed o he anda d a e ob ained f om independen da a, hich i con i en i h o heo e ical de i a ion .

6. Discussion

In hi pape , he a mp o ic di ib ion of ke nel-ba ed nonpa ame ic eg e ion e ima o de ign de c ibed in (A1.1) and (A1.2), fi ed eq all paced de ign de c ibed in (A1*), and ome ca e l ing be w een hem. The p opo ed e l co ld al o be e ended o mo e complica ed ca e , ch a panel da a w he e ob e a ion fo diffe en bjec a e ob ained a a e ie of common ime poin d ing a longi dinal follo w p. If con ide ing andom de ign, he den i of he j h ob e a ion ime T_j co ld be a med o be $f_j(t)$, hen he e l a e eadil applied o hi ca e w i h app op ia e modifica ion w i h e pec o he diffe en ma ginal den i ie .

The gene al a mp o ic di ib ion e l in ni a ia e and bi a ia e moo hing e ing a e applied o he ke nel-ba ed e ima o of he mean and co a iance f nc ion w hich ield a mp o ic no mal di ib ion of he e e ima o . To he be of o kno w ledge, he e a e no a mp o ic di ib ion e l a ailable in li e a e fo nonpa ame ic e ima o of co a iance f nc ion ob ained f om ob e ed noi longi dinal o f nc ional da a. Thi p o ide heo e ical ba i and p ac ical g idance fo he nonpa ame ic anal i of f nc ional o longi dinal da a w i h impo an po en ial applica ion ha a e ba ed on he a mp o ic di ib ion . Fo e ample, a mp o ic confidence band o egion fo he eg e ion c e o he co a iance face can be con c ed ba ed on hei a mp o ic di ib ion . Since, d e o hei hea comp a ional load, commonl ed p oced e (ch a c o - alida ion) fo band w id h elec ion in w o-dimen ional e ing a e no fea ible, one impo an e ea ch p oblem i o eek efficien app oache fo choo ing ch moo hing pa ame e . Al o f nc ional p incipal componen anal i , an inc ea ingl pop la ool fo f nc ional da a anal i , i ba ed on eigen-decompo i ion of he e ima ed co a iance f nc ion. Th , he infl ence of he a mp o ic p ope ie of co a iance e ima o on he e ima ed eigenf nc ion i ano he po en ial e ea ch of in e e .

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