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Penalized spline models for functional principal component analysis

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Summary. We propose an iterative estimation procedure for performing functional principal component analysis. The procedure aims at functional or longitudinal data where the repeated measurements from the same subject are correlated. An increasingly popular smoothing approach, penalized spline regression, is used to represent the mean function. This allows straightforward incorporation of covariates and simple implementation of approximate inference procedures for coefficients. For the handling of the within-subject correlation, we develop an iterative procedure which reduces the dependence between the repeated measurements that are made for the same subject. The resulting data after iteration are theoretically shown to be asymptotically equivalent (in probability) to a set of independent data. This suggests that the general theory of penalized spline regression that has been developed for independent data can also be applied to functional data. The effectiveness of the proposed procedure is demonstrated via a simulation study and an application to yeast cell cycle gene expression data.

Keywords: Asymptotics; Functional data; Penalized spline regression; Principal components; Smoothing; Within-subject correlation

1. Introduction

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$$G(s\ t) = \sum_{k=}^{\infty} \lambda_k \phi_k(s) \phi_k(t) \qquad t\ s \in \mathcal{T}.$$

i

$$X_i(t) = \mu(t) + \sum_{k=}^{\infty} \xi_{ik} \phi_k(t) \qquad t \in \mathcal{T}$$

$\mu(t)$ fi

$$\xi_{ik} = \int_{\mathcal{T}} \{X_i(t) - \mu(t)\} \phi_k(t) \quad t$$

$E(\xi_{ik}) = \lambda_k$ $\Sigma_k \lambda_k < \infty$

$\lambda \geqslant \lambda \geqslant$.

ε_{ij} $\sigma\ (t_{ij})$

fl $\sigma\ (t)$ ∞

$\mathcal{T}$ $<$ $_{t\in\mathcal{T}}\{\sigma\ (t)\} \leqslant$ $_{t\in\mathcal{T}}\{\sigma\ (t)\} < \infty$ Y_{ij} j $X_i(\cdot)$

t_{ij} ε_{ij} ξ_{ik} $i =$ n $j =$ n_i $k =$

n_i i

$$\begin{aligned} Y_{ij} &= X_i(t_{ij}) + \varepsilon_{ij} \\ &= \mu(t_{ij}) + \sum_{k=}^{\infty} \xi_{ik} \phi_k(t_{ij}) + \varepsilon_{ij} \qquad t_{ij} \in \mathcal{T} \end{aligned} \qquad (\ )$$

$E(\varepsilon_{ij}) =$ $E(\varepsilon_{ij}) = \sigma\ (t_{ij})$

2.2. Estimation of mean function using penalized spline regression

$\mu(t)$ $\mu(t)$

fl fi

.

$\mathbf{Y}_i = (Y_i$ $Y_{in_i})$ $\mathbf{T}_i = (t_i$ $t_{in_i})$ $B_q(t) = (B_q\ (t)$ $B_{qq}(t))$

q t $\mu(t)$

$\mu(t)$ $B_q\ (t)\boldsymbol{\beta}$ $\boldsymbol{\beta} =$

$(\beta$ $\beta_q)$ fi λ^* $\mathbf{D}$

fi $\mathbf{B}_{qi} = (B_q(t_i\ )$ $B_q(t_{in_i}))$ $n_i \times q$

$\mathbf{T}_i$ fi $\boldsymbol{\beta}$

$$\sum_{i=}^{n} \|\mathbf{Y}_i - \mathbf{B}_{qi}\boldsymbol{\beta}\| \ + \lambda^* \boldsymbol{\beta}\ \ \mathbf{D}\boldsymbol{\beta} \qquad (\ )$$

$\lambda^* \boldsymbol{\beta}\ \ \mathbf{D}\boldsymbol{\beta}$

p $B_q(t) =$

$(\ \ t$ $t^p\ (t - \kappa\ )_+^p$ $(t - \kappa_k)_+^p)$ κ κ_k $q = p + k +$

$(x)_+ =$ (x) $\mathbf{D}$ $(\mathbf{0}_{(p+\ )\times(p+\ )}$
$\mathbf{I}_{k\times k})$ $\mathbf{0}_{p\times p}$ $p \times p$ $\mathbf{I}_{p\times p}$ $p \times p$

B

κ_j

. . B
fi

$(x)_+ =$ (x) $\{\lambda_k \ \phi_k\}_{k\geqslant}$ $\{\lambda_k \ \phi_k\}_{k\geqslant}$

$$\int_{\mathcal{T}} G(s\ \ t)\,\phi_k(s)\quad s = \lambda_k\,\phi_k(t) \qquad (\ )$$

$\{\phi_k\}_{k\geqslant}$.

fi

$$\xi_{ik} = \int \{X_i(t) - \mu_{g(i)}(t)\}\,\phi_k(t)\quad t$$

$$\xi_{ik} = \sum_{j=}^{n_i} \{Y_{ij} - \mu(t_{ij})\}\,\phi_k(t_{ij})(t_{ij} - t_{i\ \ j-}\ ). \qquad (\ )$$

K

. $\boldsymbol{\mu}_i =$

$(\mu(t_i\ )\quad \mu(t_{in_i}))$ $\boldsymbol{\phi}_{ik} = (\phi_k(t_i\ )\quad \phi_k(t_{in_i}))$ $\Sigma_i =$ $\{\sigma\ (t_i\ )\quad \sigma\ (t_{in_i})\}$

ε_{ij} K

$$(K) \propto \sum_{i=}^{n} \left\{ -- \left(\mathbf{Y}_i - \boldsymbol{\mu}_i - \sum_{k=}^{K} \xi_{ik}\boldsymbol{\phi}_{ik} \right)\ \Sigma_i^{-} \left(\mathbf{Y}_i - \boldsymbol{\mu}_i - \sum_{k=}^{K} \xi_{ik}\boldsymbol{\phi}_{ik} \right) \right\} + K \qquad (\ )$$

K

3. I e a i e enali ed line ing fo i hin- bjec mea emen co ela ion

fi

$\mu_g(t)$ 1

.

.

fi

fl

$\mu^{( )}$

$l =$

1 $\mu^{(l)}$ l

$G^{(l)}$

$\sigma\ (t)$

2 $\phi_k^{(l)}$ $\lambda_k^{(l)}$

3

$\sigma\ (t)$

4 $\xi_{ik}^{(l)}$

5 i j fi

$$Y_{ij}^{*} = Y_{ij} - \sum_{i=}^{\infty} \xi_{ik}\, \phi_k(t_{ij}).$$

Y_{ij}^{*}

$$Y_{ij}^{*(l)} = Y_{ij} - \sum_{k=}^{K^{(l)}} \xi_{ik}^{(l)}\, \phi_k^{(l)}(t_{ij}) \qquad (\ )$$

$K^{(l)}$

6 Y_{ij} $Y_{ij}^{*(l)}$

$\mu^{(l+\ )}$

$\mu^{(l)}$ $\mu^{(l+\ )}$ fi

$$_l = \int_{\mathcal{T}} \{\mu^{(l+\ )}(t) - \mu^{(l)}(t)\} \quad t \Big/ \int_{\mathcal{T}} \mu^{(l)}(t) \quad t. \qquad (\ )$$

$\mu^{( )}$

1.

. ξ_{ik}^{P} ξ_{ik}

$\mathbf{Y}_i =$

2.

.

fi fi $\mathbf{a}$ $\boldsymbol{\beta}$ $\mathbf{a} \in \Re^q$

$\mathbf{Y}_i$ $\mathbf{Y}_i^* = (Y_i^* \quad Y_{in_i}^*)$

$\boldsymbol{\beta}$

$$\boldsymbol{\beta} = \left(\sum_{i=}^{n} \mathbf{B}_{qi} \mathbf{B}_{qi} + \lambda^* \mathbf{D} \right)^{-} \sum_{i=}^{n} \mathbf{B}_{qi} \mathbf{Y}_i^* .$$

$(Y_{ij}^* \; Y_{il}^*) = \delta_{jl} \sigma \; (t_{ij})$ $\sigma \; (\cdot)$

$\delta_{kl} =$ $k = l$ $\delta_{kl} =$ $\mathbf{R}_i =$ $\{\sigma \; (t_i \;) \quad \sigma \; (t_{in_i})\}$

$\Sigma_{\boldsymbol{\beta}}$ $\boldsymbol{\beta}$

$$\begin{aligned} \Sigma_{\boldsymbol{\beta}} &= \quad (\boldsymbol{\beta} \; \boldsymbol{\beta}) \\ &= \left(\sum_{i=}^{n} \mathbf{B}_{qi} \mathbf{B}_{qi} + \lambda^* \mathbf{D} \right)^{-} \left(\sum_{i=}^{n} \mathbf{B}_{qi} \mathbf{R}_i \mathbf{B}_{qi} \right) \left(\sum_{i=}^{n} \mathbf{B}_{qi} \mathbf{B}_{qi} + \lambda^* \mathbf{D} \right)^{-} . \end{aligned}$$

$(\; - \alpha)$ fi $\mathbf{a}$ $\boldsymbol{\beta}$

$$\mathbf{a} \; \boldsymbol{\beta} \pm \Phi(\; - \alpha / \;)(\mathbf{a} \; \Sigma_{\boldsymbol{\beta}} \mathbf{a})^{\;/\;} \qquad (\quad)$$

$\Sigma_{\boldsymbol{\beta}}$

fi $\Phi(\cdot)$

fi $\mathbf{B}_{qi}$

.

3.2. Theoretical properties of iterative penalized splines

$g(x \; t)$

$Y(t)$ $g \; (y \quad y \quad t \quad t \;)$ $(Y(t \;) \; Y(t \;))$

$\mu^{(\;)}$

$\mu^{(\;)}$

fi

$X(t)$

fi X

fi $\{Y_{ij}^{*(\;)}\}$

$\xi_{ik}^{(\;)}$ Y_{ij}^* j

1.
$g(x\ t)$ $g\ (x\ \ x\ \ t\ \ t\ )$

$$\leqslant k\leqslant K\ |\xi_{ik}^{(\ )}-\xi_{ik}| \to \qquad (\ \ )$$

$$\leqslant j\leqslant n_i\ |Y_{ij}^{*(\ )}-Y_{ij}^{*}| \to\ . \qquad (\ \ )$$

$\{Y_{ij}^{*}\}$ $\{Y_{ij}^{*(\ )}\}$ $\leqslant j\leqslant n_i\ |Y_{ij}^{*(\ )}-Y_{ij}^{*}|=O_p(\theta_{in})$
θ_{in} fi $O_p(\cdot)$
i $\{Y_{ij}^{*}\}$ $Y_{ij}^{*(\ )}$

G

2.
$g(x\ t)$ $g\ (x\ \ x\ \ t\ \ t\ )$

$$t\in\mathcal{T}\ |\mu(t)-\mu(t)| \to$$
$$s\ t\in\mathcal{T}\ |G(s\ \ t)-G(s\ \ t)| \to\ . \qquad (\ \ )$$

3.
μ $O_p(0(t)$

ξ_{ik}

$\mathcal{N}(\ \ \lambda_k)$ ξ_{ik} $\mathcal{N}\{(\lambda_k/\ \ )^{\ /\ }\ \lambda_k/\ \ \}$ –

$\mathcal{N}\{-(\lambda_k/\ \ )^{\ /\ }\ \lambda_k/\ \ \}$ – $\{c\ \ \ \ c\ \ \}$

$c\ \ =$ $c\ \ =$ $s_i = c_i + e_i$ e_i

$\mathcal{N}(\ \ \ .\ \ )$ $s_i =$ $s_i <$ $s_i =$ $s_i >$

$\{\ \ \ \ \}$

$\{s\ \ \ \ s\ \ \}$

$\{\ \ \ \ \}$

$\mu(t)$

$\mu(t)$

$\mu^{(\ )}$

$\mu^{(\ )}$

$$K\ \ (x) = -(\ \ -x\ \ )\mathbf{1}_{\ -\ }\ \ (x)$$

$$K\ \ (x\ \ y) = \text{—}(\ \ -x\ \ )(\ \ -y\ \ )\mathbf{1}_{\ -\ }\ \ (x)\mathbf{1}_{\ -\ }\ \ (y)$$

$\mathbf{1}_A(x) =$ $x \in A$ $\mathbf{1}_A(x) =$ A

$p =$

$\mu(t)$

$$\int\ \ E\ \{\mu(t)-\mu(t)\}\ \ \ t = \int\ \ \mu(t) - E\{\mu(t)\}\ \ \ t + \int\ \ E\{\mu(t)\} - \mu(t)\ \ \ t.$$

K $X_i^K(t) =$

$\mu(t) + \Sigma_{k=}^{K}\ \xi_{ik}\,\phi_k(t)$ ξ_{ik}

Table 1. Simulation results for comparing mean estimates obtained by methods 1–4 from 100 Monte Carlo runs with $n = 100$ random trajectories per sample

D						
	B A	A		B A	A	

λ^* K

fi h_μ $\mu^{(\ )}(t)$

L

$\int \{\mu^{(\ )}(t\ h_\mu) - \mu(t)\}\ t$

h_G h_V λ^*

$X(t)$ $\varepsilon(t)$ $\mu(t)$

L K fi

h_μ h_G h_V

fi

λ^*

K

λ^*

ξ_{ik}

fi

$K =$

fi

–

X_i

$$= \sum_{i=}^{n} \int \{X_i(t) - X_i^K(t)\} \quad t/n$$

$X_i^K(t) = \mu(t) + \Sigma_{k=}^{K} \xi_{ik}\,\phi_k(t)$

5. A lica ion o ea cell c cle gene e e ion da a

.

α

fi

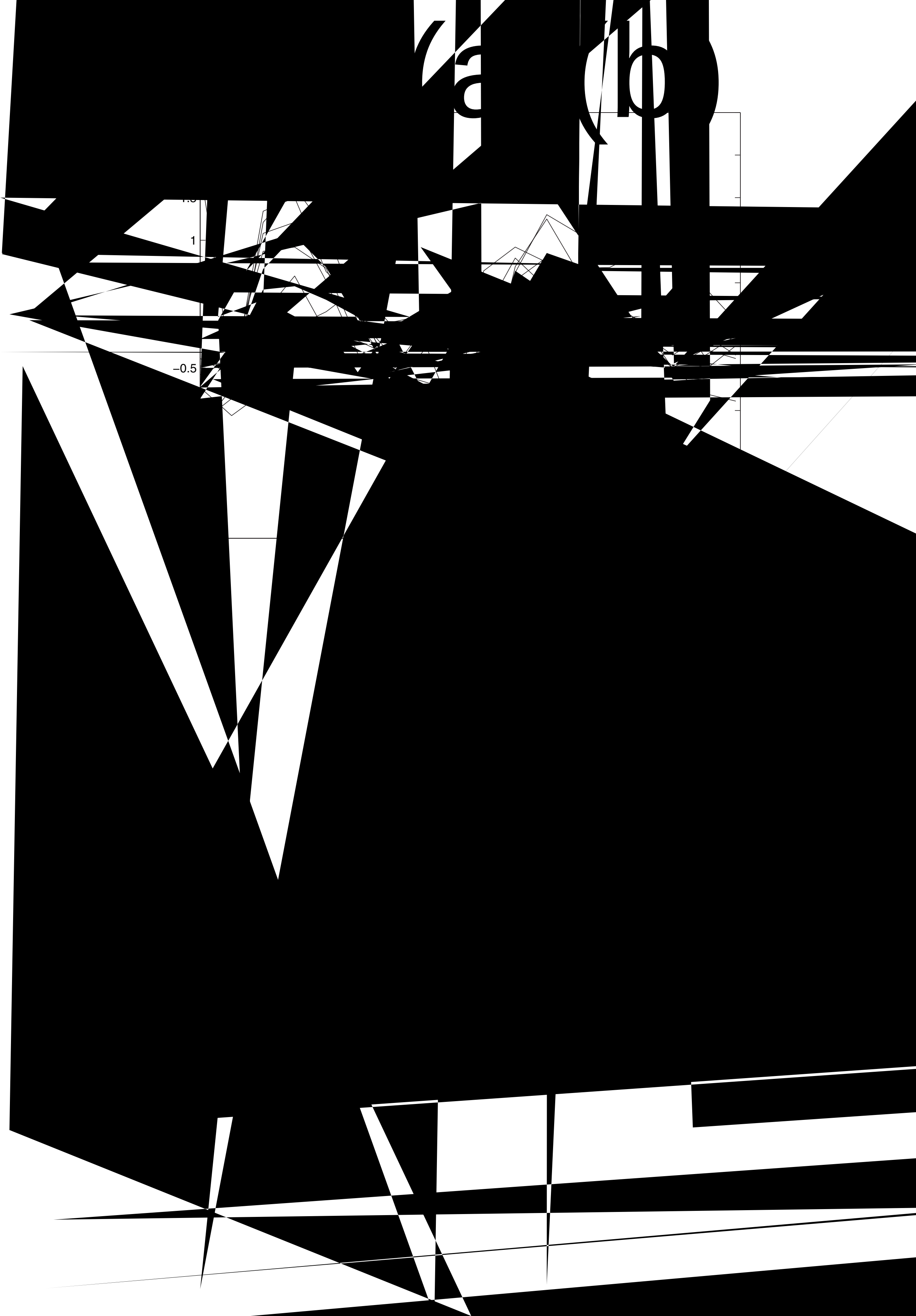
fi fi

fi

$\mu(t) \approx B_q(t)\boldsymbol{\beta}$

fi

h_μ λ^*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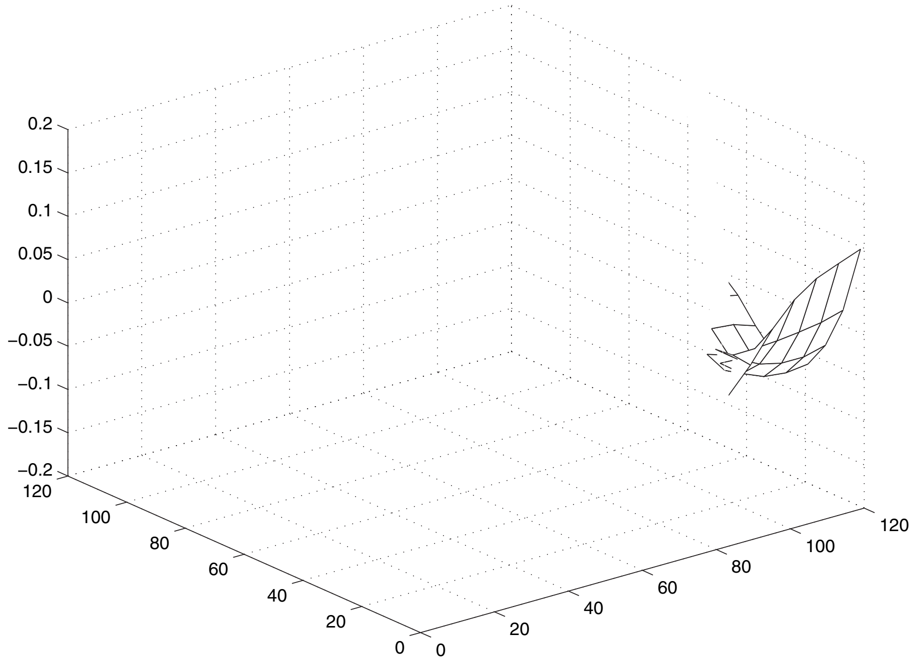

(a) Smooth estimate of the covariance surface and (b) three eigenfunctions obtained using the IPSprocedure proposed, for the G1 phase yeast cell cycle gene expression profiles

Time (minutes) (a)

fi

$$X_i(t) = \mu(t) + \sum_{k=}^{K} \xi_{ik}\, \phi_k(t)$$

ξ_{ik}

fi fi

fi

$$= \frac{}{n} \sum_{i=}^{n} \sum_{j=}^{n_i} \frac{\{Y_{ij} - Y_i(t_{ij})\}}{n_i}.$$

$=$. $=$

.

6. Concl ding ema k

fi

fi

Ackno ledgemen

A endi A

A.1. Assumptions and notation

fi $\mu(t)$

$$\sum_{i=}^{n} \sum_{l=}^{n_i} K\left(\frac{t_{ij} - t}{h_\mu}\right) \{Y_{ij} - \beta \;\; - \beta \;\; (t - t_{ij})\} \qquad (\quad)$$

β β $\mu^{(\,)}(t) = \beta\;\;(t)$

$g =$

K K fi

$h_\mu = h_\mu(n)$ $h_G = h_G(n)$ $h_V = h_V(n)$ $\mu^{(\,)}$ $G^{(\,)}$

$V^{(\ )}$ $n \to \infty$

$h_\mu \to$ $h_V \to$ $nh_\mu \to \infty$ $nh_V \to \infty$ $nh_\mu < \infty$ $nh_V < \infty$
$h_G \to$ $nh_G \to \infty$ $nh_G < \infty$

$\{t_{ij}\}_{i= \ \dots\ n\ j= \ \dots\ n_i}$
$a_X \leqslant t_{(\ )} \leqslant \ \leqslant t_{(N_n)} \leqslant b_X$ $\Delta_n =$ $\{t_{(k)} - t_{(k-\ )}\ k =$ $N +\ \}$ $N_n = \Sigma_{i=}^n n_i$
$\mathcal{T} = a_X\ b_X$ $t_{(\ )} = a_X$ $t_{(N+\ )} = b_X$ i t_{ij}
$\Delta_{in} =$ $\{t_{ij} - t_{i\,j-}\ j =$ $n_i +\ \}$ $\Delta_n^* =$ $\{\Delta_{in}\ i =$ $n\}$
$t_i = a_X$ $t_{i\,n_i+} = b_X$ $\bar{n} = n^- \Sigma_{i=}^n n_i$
$\mathcal{T}$

$\Delta_n = O(\ \{n^{-\ /\ } h_\mu^- \ n^{-\ /\ } h_V^- \ n^{-\ /\ } h_G^- \})$
$\bar{n} \to \infty$ $\{n_i\ i =$ $n\} \leqslant C\bar{n}$ $C >$ $\Delta_n^* = O(\ /\bar{n})$ $n \to \infty$

$K\ (u)$ $K\ (u\ v)$ $\kappa\ (t) = \int (-\ ut)\, K\ (u)\ u$ $\kappa\ (t\ s) =$
$\int \{-(\ ut +\ vs)\}\, K\ (u\ v)\ u\ v$

$\kappa\ (t)$ $\int |\kappa\ (t)|\ t < \infty$
$\kappa\ (t\ s)$ $\int\int |\kappa\ (t\ s)|\ t\ s < \infty$

$Y(t)$ $t \in \mathcal{T}$

${}_{t\in\mathcal{T}}\ E\{Y\ (t)\} < \infty$

fi $f \otimes g = \langle f\ h\rangle y$ $f\ h \in H$
H $F \equiv \sigma\ (H)$ $\langle T\ \ T\ \rangle_F =\ (T\ T^*) = \Sigma_j \langle T\ u_j\ T\ u_j\rangle_H$
$\|T\|_F = \langle T\ T\rangle_F$ $T\ \ T\ \ T \in F$ $\{u_j\ j \geqslant\ \}$ H
G **G** G G $\mathbf{G}(f) = \int_{\mathcal{T}} G(s\ t)\, f(s)\ s$
$\mathbf{G}(f) = \int_{\mathcal{T}} G(s\ t)\, f(s)\ s$
$\mathcal{I}_i = \{j\ \lambda_j = \lambda_i\}$ $\mathcal{I}' = \{i\ |\mathcal{I}_i| =\ \}$ $|\mathcal{I}_i|$ $\mathcal{I}_i$ $\mathbf{P}_j =$
$\Sigma_{k\in\mathcal{I}_j} \phi_k \otimes \phi_k$ $\mathbf{P}_j = \Sigma_{k\in\mathcal{I}_j} \phi_k \otimes \phi_k$
H $\{\phi_k\ k \in \mathcal{I}_j\}$ fi j

$$\delta_j = \text{-}\ \{|\lambda_l - \lambda_j|\ l \notin \mathcal{I}_j\} \qquad (\ )$$

$\mathbf{\Lambda}_{\delta_j} = \{z \in \mathcal{C}\ |z - \lambda_j| = \delta_j\}$ $\mathcal{C}$ **G**
G **R** **R** $\mathbf{R}(z) = (\mathbf{G} - zI)^-$ $\mathbf{R}(z) = (\mathbf{G} - zI)^-$

$$A_{\delta_j} = \ \{\|\mathbf{R}(z)\|_F\ z \in \mathbf{\Lambda}_{\delta_j}\}. \qquad (\ )$$

$K = K(n)$ $X(t)$

$$X_i(t) = \mu^{(\ )}(t) + \sum_{k=}^{K} \xi_{ik}^{(\ )} \phi_k^{(\ )}(t)$$

fi K $K = K^{(\ )}$ $\|\pi\|_\infty = \ {}_{t\in\mathcal{T}}\{|\pi(t)|\}$
$\pi(\cdot)$ $\mathcal{T}$ K
n $n \to \infty$

$K \to \infty$ $\upsilon_n = \Sigma_{k=}^K \delta_k A_{\delta_k} \|\phi_k\|_\infty / (n^{\ /\ } h_G - A_{\delta_k}) \to$
$\Sigma_{k=}^K \|\phi_k\|_\infty = o(\ \{n^{\ /\ } h_\mu\ \bar{n}^{\ /\ }\})$ $\Sigma_{k=}^K \|\phi_k\|_\infty \|\phi_k'\|_\infty = o(\bar{n})$

K $n \to \infty$
δ_k fl A_{δ_k}
G λ_k
K n $n \gg K$
fi
X

$E(\|X\|_\infty + \|X'\|_\infty) < \infty$ $E\ \{\ {}_{t\in\mathcal{T}} |X(t) - X^K(t)|\} = o(n)$ $X^K(t) = \mu(t) + \Sigma_{k=}^K \xi_{ik}\phi_k(t)$

$Y^*_{ij} = Y_{ij} - \Sigma^{\infty}_{k=} \xi_{ik}\phi_k(t)$ $Y^*_{ij} = \mu(t) + \varepsilon_{ij}$
$\mu(t)$ $\mu(t\ \beta\ q) = \Sigma^q_{l=} \beta_l b_l(t)$ p k
$b_l(t) = t^l$ $\leqslant l \leqslant p$ $b_l(t) = (t - \kappa_{l-p})^p_+$ $p +\ \leqslant l \leqslant k$ $q = p + k +$
$\mu(t)$

Y_{ij} Y^*_{ij} $\tilde{\mu}(t)$ fi
$\mu(t)$

$b(t|\kappa) = (t - \kappa)^p_+$
$\kappa \in \mathcal{T}$ $b_l(t) = b(t|\kappa_{l-p})$ $l \geqslant p +$ $a(t)$ κ_j
$j \leqslant q - p$ $t \in \mathcal{T}$ q
fi $\tilde{\mu}$

∞ fi p $n \to \infty$ $a(t)$
∞ $\mathcal{T}$

$b(t|\kappa)$ fi ψ

$$\psi(u\ v) = \int_{\mathcal{T}} b(t|u)\, b(t|v)\ \ v$$

α $\psi\alpha$ fi

$$(\psi\alpha)(u) = \int_{\mathcal{T}} \psi(u\ v)\, \alpha(v)\ \ t.$$

$\mu^*(t) = \mu(t) - \Sigma^p_{l=}\ b_l(t)$
fi β^*

$$\mu^*(t) = \int_{\mathcal{T}} \beta^*(s)\, b(t|s)\, a(s)\ \ s$$

$t \in \mathcal{T}$

${}_{t\in\mathcal{T}}\{\int_{\mathcal{T}} b(t|s)\ \ s\} < \infty$ ψ β^* $\int_{\mathcal{T}} \beta^*(t)$
$t < \infty$

$\{\rho_j\}_{j=\ \dots}$ $\{\psi_j\}_{j=\ \dots}$
ψ

$\Sigma^{\infty}_{j=} |\int_{\mathcal{T}} \beta^*(t)\, \psi_j(t)\ \ t| + \Sigma^{\infty}_{j=} \sqrt{\{\rho_j}$ $(j)\} < \infty$ $\lambda^* \to$ fi $n \to n$
$n^{-\ /}$ $\Sigma^{\infty}_{j=} \sqrt{\{\rho_j}$ $(j)\}/(\rho_j + \lambda^*) \to$ $\lambda^* = \lambda^*(n)$
$\tilde{\mu}(t)$

$g(y\ t)$ $Y(t)$ $g\ (y\ \ y\ \ t\ \ t\ )$ $(Y(t\ )\ Y(t\ ))$

fi

.

t_{ij} .

ν l $\leqslant \nu < l$

$(\ ^l/\ t^l)\, g(y\ t)$ $\Re \times \mathcal{T}$

K $(\nu\ l)$ $\int u^q K\ (u)\ \ u$ $(-\ )^\nu \nu$ $q = \nu$
$q = l$ $K\ \ \Re \to \Re$

K $(\nu\ l)$ $\|K\ \| = \int K\ (u)\ \ u < \infty$

fi .
$q \geqslant$ $(\psi_p)_{p=\ \dots\ q}$ $\psi_p\ \ \Re\ \to \Re$

ψ_p $\mathcal{T}\times\Re$

$(\ ^l/\ t^l)\,\psi_p(t\ \ x)$ $(t\ \ x)$ $\mathcal{T}\times\Re$

$_{t\in\mathcal{T}}\{\int\psi_p(t\ \ x)\,g(x\ \ t)\ \ x\ \ t\}<\infty.$

$h_\mu=h_\mu(n)$

$h_\mu\to\ \ nh_\mu^{\nu+}\ \to\infty\ \ nh_\mu^{l+}\ <\infty\ \ \Delta_n=O\{\ /(n\ ^/\ h_\mu^{\nu+}\ )\}$ $\{n_i\ \ i=\ \ \ n\}\leqslant C\bar{n}$ $n\to\infty$

fi

$$\Psi_{pn}=\Psi_{pn}(t)$$
$$=\frac{}{nh_\mu^{\nu+}}\sum_{i=}^{n}\frac{}{\bar{n}}\sum_{j=}^{n_i}\psi_p(t_{ij}\ \ Y_{ij})\,K\left(\frac{t-t_{ij}}{h_\mu}\right)\qquad p=\qquad q$$

$$\mu_p=\mu_p(t)$$
$$=\frac{\nu}{t^\nu}\int\psi_p(t\ \ x)\,g(x\ \ t)\ \ x\qquad p=\qquad q.$$

A.2. Auxiliary results and proofs of main theorems

1. $\tau_{pn}=\ \ _{t\in\mathcal{T}}|\Psi_{pn}(t)-\mu_p|=O_p\{\ /(n\ ^/\ h_\mu^{\nu+}\ )\}$

. fi

t_{ij}

.

fi

2. $h_\mu\ \ h_G$ h_V

$\mu^{(\ )}(t)$ $G^{(\ )}(s\ \ t)$ $V^{(\ )}(t)$

$g(y\ \ t)$ $g\ (y\ \ \ y\ \ \ t\ \ \ t\ )$

$$_{t\in\mathcal{T}}|\mu^{(\ )}(t)-\mu(t)|=O_p\left(\frac{}{n\ ^/\ h_\mu}\right)$$
$$_{s\,t\in\mathcal{T}}|G^{(\ )}(s\ \ t)-G(s\ \ t)|=O_p\left(\frac{}{n\ ^/\ h_G}\right).\qquad(\ \ )$$

λ_k ϕ_k

$$_{t\in\mathcal{T}}|\phi_k^{(\ )}(t)-\phi_k(t)|=O_p\left(\frac{\delta_k A_{\delta_k}}{n\ ^/\ h_G-A_{\delta_k}}\right)$$
$$\lambda_k^{(\ )}-\lambda_k=O_p\Big(\ \ \ \delta_k A_{\delta_k/}\ _{G\|A}$$

3. λ^*

$\tilde{\mu}(t)$ $g(y\ t)$

$$_{t\in\mathcal{T}} |\mu^*(t)-\mu(t)| = O_p(\omega_n) \qquad \omega_n = \frac{}{n\ /}\sum_{j=}^{\infty}\frac{\sqrt{\{\rho_j\quad (j)\}}}{\rho_j+\lambda^*}+\sum_{j=}^{\infty}\frac{\lambda^*|\int_{\mathcal{T}}\beta^*(t)\,\psi_j(t)\quad t|}{\rho_j+\lambda^*}. \qquad (\ \ )$$

$\|X_i\|_L = \{\int_{\mathcal{T}} X_i(t) \quad t\}^{/}$ c c i k

$$\underset{\leqslant k \leqslant K}{} |\tilde{\eta}_{ij} - \xi_{ik}| \leqslant \underset{\leqslant k \leqslant K}{} \{\|(X_i + \mu)'\phi_k + (X_i + \mu)\phi_k'\|_\infty \Delta_n^*\}$$
$$\leqslant \underset{\leqslant k \leqslant K}{} (\|X_i\|_\infty \|\phi_k\|_\infty + \|X_i'\|_\infty \|\phi_k\| + c \ \|\phi_k\|_\infty + c \ \|\phi_k\|_\infty)\Delta_n^*$$
$$\leqslant (c \ \|X_i\|_\infty + c \ \|X_i'\|_\infty + c \ ) \underset{\leqslant k \leqslant K}{} (\|\phi_k'\|_\infty \Delta_n^*) \to \qquad (\ \ )$$

c c i k

fi

$$\sum_{k=}^{K} |\tau_{ik}| \, \|\phi_k\|_\infty \to \ .$$

$$|\tau_{ik}| \leqslant |\tilde{\tau}_{ik}| + \sum_{j=}^{n_i} |\varepsilon_{ij}| \, |\phi_k^{(\ )}(t_{ij}) - \phi_k(t_{ij})| (t_{ij} - t_{i\ j-}\ ).$$

$E(\tilde{\tau}_{ik}) =$

$$(\tilde{\tau}_{ik}) = \sum_{j=}^{n_i} \sigma \ (t_{ij}) \, \phi_k(t_{ij}) (t_{ij} - t_{i\ j-}\ )$$
$$\leqslant \underset{t \in \mathcal{T}}{} \{\sigma \ (t)(\ + \ \|\phi_k\|_\infty \|\phi_k'\|_\infty \Delta_n^*)\Delta_n^*\}$$
$$\leqslant \underset{t \in \mathcal{T}}{} \{\sigma \ (t) \Delta_n^*\}$$

$$\sum_{k=}^{K} |\tilde{\tau}_{ik}| \|\phi_k\|_\infty \leqslant \underset{t \in \mathcal{T}}{} \{\sigma \ (t)\Delta_n^*\}^{\ /} \sum_{k=}^{K} \|\phi_k\|_\infty \to$$

$$\sum_{k=}^{K} \sum_{j=}^{n_i} |\varepsilon_{ij}| |\phi_k^{(\ )}(t_{ij}) - \phi_k(t_{ij})| (t_{ij} - t_{i\ j-}\ ) \|\phi_k\|_\infty \leqslant \upsilon_n \sum_{j=}^{n_i} |\varepsilon_{ij}| (t_{ij} - t_{i\ j-}\ )$$

$$E\left\{ \sum_{j=}^{n_i} |\varepsilon_{ij}| (t_{ij} - t_{i\ j-}\ ) \right\} \leqslant |\mathcal{T}| \underset{t \in \mathcal{T}}{} \{\sigma(t)\}$$

$\Sigma_{j=}^{n_i} |\varepsilon_{ij}| (t_{ij} - t_{i\ j-}\ ) = O_p(\ )$

$$\sum_{k=}^{K} |\tau_{ik}| \|\phi_k\|_\infty \to \ .$$

fi

$$\underset{t \in \mathcal{T}}{} \left| \sum_{k=}^{K} \xi_{ik}^{(\ )} \phi_k^{(\ )}(t) - \sum_{k=}^{\infty} \xi_{ik} \phi_k(t) \right| \leqslant \underset{t \in \mathcal{T}}{} \left| \sum_{k=}^{K} \{\xi_{ik}^{(\ )} \phi_k^{(\ )}(t) - \xi_{ik} \phi_k(t)\} \right| + \underset{t \in \mathcal{T}}{} \left| \sum_{k=K+}^{\infty} \xi_{ik} \phi_k(t) \right| \to \ . \qquad (\ \ )$$

$K \to \infty$ $n \to \infty$ fi

$$_{t\in\mathcal{T}}\left|\sum_{k=}^{K}\{\xi_{ik}^{(\,)}\phi_k^{(\,)}(t)-\xi_{ik}\phi_k(t)\}\right|\leqslant\sum_{k=}^{K}|\xi_{ik}^{(\,)}-\xi_{ik}|(\|\phi_k\|_\infty+\tilde{\upsilon}_n)+\left|\sum_{k=}^{K}\xi_{ik}\{\phi_k^{(\,)}(t)-\phi_k(t)\}\right|$$
$$\equiv Q\ \ (n)+Q\ \ (n).$$

$$E|Q\ \ (n)|\leqslant\sum_{k=}^{K}\delta_k A_{\delta_k}E|\xi_{ik}|/(n^{\ /\ }h_G-A_{\delta_k})\leqslant\sum_{k=}^{K}\delta_k A_{\delta_k}\lambda_k^{\ /\ }/(n^{\ /\ }h_G-A_{\delta_k})$$

$\lambda_k\to$ $E|Q\ \ (n)|=O(\upsilon_n)$ $Q\ \ (n)=O_p(\upsilon_n)$

$$Q\ \ (n)\leqslant\ \sum_{k=}^{K}|\xi_{ik}^{(\,)}-\xi_{ik}|\|\phi_k\|_\infty$$

n

$$\sum_{k=}^{K}|\xi_{ik}^{(\,)}-\xi_{ik}|\|\phi_k\|_\infty\leqslant\sum_{k=}^{K}|\eta_{ik}-\tilde{\eta}_{ik}|\|\phi_k\|_\infty+\sum_{k=}^{K}|\tilde{\eta}_{ik}-\xi_{ik}|\|\phi_k\|_\infty+\sum_{k=}^{K}|\tau_{ik}|\|\phi_k\|_\infty.\qquad(\ \ )$$

fi

$$\{c\ \ (\|X_i\|_{L}\ +\|X_i\|_\infty\|X_i'\|_\infty\Delta_n^*)+c\ \}\upsilon_n+\left(\ +\sum_{k=}^{K}\|\phi_k\|_\infty\|\phi_k'\|_\infty\Delta_n^*\right)\frac{\sum_{k=}^{K}\|\phi_k\|_\infty}{n^{\ /\ }h_\mu}\to\ .$$

$$(c\ \|X_i\|_\infty+c\ \|X_i'\|_\infty+c\ )\sum_{k=}^{K}\|\phi_k\|_\infty\|\phi_k'\|_\infty\Delta_n^*\to\ .$$

$$\sum_{k=}^{K}|\tau_{ik}|\|\phi_k\|_\infty\to\ .$$

$_{\leqslant j\leqslant n_i}|Y_{ij}^*-Y_{ij}^{*(\,)}|=O_p(\theta_{in})$ $O_p(\cdot)$

i θ_{in} fi $\mu(t)$ $G(s\ \ t)$

A.2.2. 2

$\tilde{\mu}(t)$ fi

Y_{ij}^* $\mu(t)$ $Y_{ij}^{*(\,)}$ $\tilde{G}$

$\tilde{\mu}(t)$ G

$\mu(t)$ fi

$Y_{ij}^{*(\,)}=Y_{ij}^*+O_p(\theta_{in})$ $O_p(\cdot)$ j

$_{t\in\mathcal{T}}|\mu(t)-\tilde{\mu}(t)|=O_p(\bar{\theta}_n)$ $_{s\ t\in\mathcal{T}}|G(s\ \ t)-\tilde{G}(s\ \ t)|=O_p(\bar{\theta}_n)$ $\bar{\theta}_n=\Sigma_{i=}^n\theta_{in}$

$$E(\|X\|_\infty\|X'\|_\infty)\leqslant\{E(\|X\|_\infty)E(\|X'\|_\infty)\}^{\ /\ }<\infty$$
$$E\left\{\sum_{j=}^{n_i}|\varepsilon_{ij}|(t_{ij}-t_{i\,j-\ })\right\}\leqslant|\mathcal{T}|\ _{t\in\mathcal{T}}\{\sigma(t)\}<\infty$$

$$E\left\{\sum_{k=}^{K}\delta_k A_{\delta_k}|\xi_{ik}|/(n^{\ /\ }h_G-A_{\delta_k})\right\}\leqslant\sum_{k=}^{K}\delta_k A_{\delta_k}\lambda_k^{\ /\ }/(n^{\ /\ }h_G-A_{\delta_k})\leqslant\upsilon_n$$

$$\bar{\theta}_n = O_p(\theta_n^*) \to$$

θ_n^* fi

$\mu(t)$ $G(t)$

$$_{t\in\mathcal{T}} |\mu(t) - \mu(t)| = O_p(\omega_n + \theta_n^*)$$

$$_{s\, t\in\mathcal{T}} |G(s\ \ t) - G(s\ \ t)| = O_p\left(\omega_n + \theta_n^* + \frac{}{n\ /\ h_G}\right) \quad (\ \ )$$

ω_n θ_n^* h_G

Refe ence.

B **47**

51

55

fi
. *A* . . *A* . **100**

fi . *A* . . *A* . **97**

B **59**

. *A* . .
A . **100**