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CONTRIBUTED ARTICLE

Simplex Memory Neural Networks

JINWEN MA

Institute of Mathematics, Shantou University

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Abstract—*The biological neural network—simplex memory neural network—is proposed to describe the mechanisms of pattern memory in the brain. A mathematical model of the simplex memory neural network is constructed to memorize any binary pattern with content-addressable memory function. Under Hebbian learning rule, the new network has some important functions in accord with the learning and memory behaviors of the brain. Copyright © 1996 Elsevier Science Ltd.*

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1. INTRODUCTION

A fundamental question about understanding the memory mechanisms of the brain is "where and how is the information memorized in the brain." This question has not been structurally solved yet. By physiological anatomy, it is clear that the brain is made of of neurons (neural cells), which are only in the excited state 1 or the quiescent state 0 as being expressed in a MP model (McCulloch & Pitts, 1942). Then we generally consider that any piece of information memorized in the brain is a binary code—A (binary) pattern is a binary code to express

wonder how a pattern can be memorized for a long term and where it is. Psychological experiments show that the memory is stored in the brain by groups individually, and the memorizing or learning behavior on a group is simply reading it repeatedly. In this way, a pattern (as one group of information) is itself probably memorized individually in a local place—a number of neurons—in the brain. This set of neurons, in fact, form a local neural network. Hence we suppose that a pattern is itself memorized in a unique local neural network of the brain which we call a simplex memory neural network. More concretely, The purpose of this article is to construct a model of SMNNs. As to associative memory, there are many researches approaching it. It can be done by means of an associative matrix or other methods. Since the structures for associative memory are built between two networks, we will not consider AM functions of the SMNN for simplicity here. Then the architecture of a SMNN will be established only by the CAM function.

2. THE MATHEMATICAL MODEL OF SMNN

In this section, a model of SMNN is constructed mathematically. Firstly, let N be a neural network of n pair-wised connected neurons, then N is uniquely defined by $(W, \boldsymbol{\theta})$ where W is an $n \times n$ zero-diagonal matrix, where element w_{ij} denotes the weight from neuron j to neuron i; $\boldsymbol{\theta}$ is a vector of dimension n, where component θ_i denotes the threshold of neuron i.

Every neuron here can be in one of two possible states, either 1 or 0. The state of neuron i at time t is denoted by $s_i(t)$. The state of the neural network at time t is the vector $\mathrm{S}(t) = (s_1(t), s_2(t), \ldots, s_n(t))^{\mathrm{T}}$.

The state of neuron i at time $(t+1)$ is computed by

$$s_i(t+1) = \mathrm{Sgn}(H_i(t)) = \begin{cases} 1 & \text{if } H_i(t) \geq 0, \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where

$$H_i(t) = \sum_{j=1}^{n} w_{ij} s_j(t) - \theta_i. \tag{2}$$

The next state of the network, i.e., $\mathrm{S}(t+1)$, is computed from the current state by performing the evaluation eqn (1) at any neuron of the network. The mode of operation is synchronous. When the symmetric condition is satisfied, i.e., $w_{ij} = w_{ji}$ for any pair $(i, j)(i \neq j)$, the network is certainly a Hopfield network (Hopfield, 1982) of the synchronous operation mode.

The state $\mathrm{S}(t)$ is called stable if

$$\mathrm{S}(t) = \mathrm{Sgn}(W S(t) - \theta), \tag{3}$$

i.e., the state of the network is not changing as a result of computation.

As a dynamic system, this kind of network may have a CAM function as Hopfield networks. The network starts in an initial state and runs with each neuron synchronously re-evaluating itself. The network enters a stable state which constitutes a stored pattern in the memory.

According to the CAM function of the networks, we introduce a mathematical definition of the SMNN of a pattern $U = (u_1, u_2, \ldots, u_n)^{\mathrm{T}}$ as follows.

Given $X = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ as a n-dim input pattern, we define

$$d(X, U) = \sum_{i=1}^{n} |x_i - u_i| \tag{4}$$

as the Hamming distance between X and U.

DEFINITION 1. *Define the t-neighborhood of U over the binary n-dim space* $\{0, 1\}^n$ *as follows:*

$$R_t(U) = \{X\colon d(X, U) \leq t, X \in \{0, 1\}^n\}. \tag{5}$$

DEFINITION 2. *As to a fixed pattern U, if a network* (W, θ) *of n neurons satisfies:*

$$E(X) = \begin{cases} U & \text{if } X \in R_t(U), \\ O & \text{otherwise} \end{cases} \tag{6}$$

where E(X) is the stable state of the network (if it exists) when the network starts in the initial state X. Then the network is called a SMNN of the pattern U with error-correcting capacity t $(t \geq 0)$.

From Definition 2, the performance of a SMNN can be understood in this way. Firstly the initial state X is inputted to the network. If the network operates at last to the stable state U, we think the network is excited, and the pattern U is retrieved, or U is perceived. If the stable state of the network at last is O, which means the network is in the quiescent state, we think the pattern U is not retrieved. By the definition of $R_t(U)$, the SMNN certainly has a CAM function.

As to a fixed pattern, if the required error-correcting capacity t satisfies:

$$0 \leq t < \left(\frac{1}{2} d(U) - 1\right). \quad \left(d(U) = \sum_{i=1}^{n} u_i\right) \tag{7}$$

where $d(U)$ is the Hamming weight of U, we now propose a symmetric SMNN of U with error-correcting capacity t (like a Hopfield network) by

$$w_{ij} = \begin{cases} 1 & u_i = u_j = 1, i \neq j, \\ -1 & u_i \neq u_j, i \neq j, \\ 0 & u_i = u_j = 0, i \neq j, \\ 0 & i = j \end{cases} \tag{8}$$

$$\theta_i = d(U) - (t+1) \tag{9}$$

for all $i, j = 1, 2, \ldots, n$.

Although the weight matrix and threshold vector are constructed mathematically, they have some biological meanings which will be discussed later. Now we prove the network by showing the weight matrix and threshold vector is a SMNN of U with error-correcting capacity t.

THEOREM *1. For any fixed pattern U, if $d(U) > 0$, and $0 \le t < \frac{1}{2}d(U) - 1$, the network (W, θ) constructed by eqns (8) and (9), is a SMNN of U with error-correcting capacity t.*

Proof. Firstly we define the one-step evolution function F from $\{0, 1\}^n$ into $\{0, 1\}^n$ of the network (W, θ) as for $X = (x_1, x_2, \ldots, x_n)^T \in \{0, 1\}^n$

$$F_i(X) = \mathrm{Sgn}\left(\sum_{j=1}^{n} w_{i,j}x_j(t) - \theta_i\right), \quad i = 1, 2, \ldots, n. \tag{10}$$

Then the state at a time $t+1$ is given by $S(t+1) = F(S(t))$. It is easy to verify that U and O are the stable states of the network, i.e., they satisfy that

$$F(U) = U \quad \text{and} \quad F(O) = O. \tag{11}$$

In fact, in view of (8), we have that when $u_i = 1$

$$w_{ij} = \begin{cases} 1 & \text{if } u_j = 1 \text{ and } i \ne j, \\ -1 & \text{if } u_j = 0 \text{ and } i \ne j, \\ 0 & \text{if } i = j \end{cases} \tag{12}$$

and when $u_i = 0$

$$w_{ij} = \begin{cases} -1 & \text{if } u_j = 1 \text{ and } i \ne j, \\ 0 & \text{if } u_j = 0 \text{ and } i \ne j, \\ 0 & \text{if } i = j. \end{cases} \tag{13}$$

Hence

$$\sum_{j=1}^{n} w_{ij}u_j = \begin{cases} d(U) - 1 & \text{if } u_i = 1, \\ -d(U) & \text{if } u_i = 0. \end{cases} \tag{14}$$

Thus, when $u_i = 1$

$$\begin{aligned} F_i(U) &= \mathrm{Sgn}((d(U) - 1) - (d(U) - (t+1))) \\ &= \mathrm{Sgn}(t) = 1 = u_i \end{aligned} \tag{15}$$

and when $u_i = 0$

$$\begin{aligned} F_i(U) &= \mathrm{Sgn}((-d(U)) - (d(U) - (t+1))) \\ &= \mathrm{Sgn}(t + 1 - 2d(U)) = 0 = u_i \end{aligned} \tag{16}$$

which implies that $F(U) = U$. On the other hand

$$\begin{aligned} F_i(O) = \mathrm{Sgn}(-\theta_i) = \mathrm{Sgn}(t + 1 - d(U)) = 0, \\ i = 1, 2, \ldots, n. \end{aligned} \tag{17}$$

Now we show that for any pattern $X = (x_1, x_2, \ldots, x_n)^T \in \{0, 1\}^n$

$$u_i = 0 \Longrightarrow F_i(X) = 0 = u_i \tag{18}$$

and

$$u_i = 1 \Longrightarrow F_i(X) = \begin{cases} 1 = u_i & \text{if } d(X, U) \le t, \\ 0 & \text{if } d(X, U) > t + 1, \\ 1 - x_i & \text{if } d(X, U) = t + 1. \end{cases} \tag{19}$$

In order to show them we introduce a variable determined from an input pattern X and a fixed pattern U as follows:

$$N^{ij}(X, U) = |\{k : x_k = i \text{ and } u_k = j\}| \quad \text{for } i, j = 0, 1, \tag{20}$$

where $|A|$ denotes the number of elements of a set A. Then for $u_i = 0$ we have that, in view of (13) and $N^{11}(X, U) \ge 0$,

$$\begin{aligned} \sum_{j=1}^{n} w_{ij}x_j - \theta_i &= -N^{11}(X, U) - d(U) + t + 1 \\ &\le t + 1 - d(U) < 0, \end{aligned} \tag{21}$$

which implies (18). Before proving (19), we note that

$$d(U) = N^{11}(X, U) + N^{01}(X, U) \tag{22}$$

and

$$d(X, U) = N^{10}(X, U) + N^{01}(X, U). \tag{23}$$

Now suppose that $u_i = 1$. Then, in view of (12),

$$\begin{aligned} \sum_{j=1}^{n} w_{ij}x_j &= -x_i + \left(\sum_{j: u_j=1} w_{ij}x_j + x_i + \sum_{j: u_j=0} w_{ij}x_j\right) \end{aligned} \tag{24}$$

$$= -x_i + N^{11}(X, U) - N^{10}(X, U)$$

in view of (22),

$$= -x_i + d(U) - N^{01}(X, U) - N^{10}(X, U)$$

in view of (23),

$$= -x_i + d(U) - d(X, U).$$

Hence

$$\begin{aligned} \sum_{j=1}^{n} w_{ij}x_j - \theta_i &= -x_i + d(U) - d(X, U) - (d(U) - (t+1)) \\ &= -x_i + t + 1 - d(X, U), \end{aligned} \tag{25}$$

which implies (19).

We now consider the stable states of the network with the initial state X:

(1) The case of $d(X, U) \leq t$: in view of (18) and (19) we have $F(X) = U$ for any pattern $X \in \{0, 1\}^n$ such that $d(X, U) \leq t$. Thus the initial state X arrives at the stable state U [cf. (11)] with only one step and we have $E(X) = U$.

(2) The case of $d(X, U) > t + 1$: in view of (18) and (19) we have $F(U) = O$ for any pattern $X \in \{0, 1\}^n$ such that $d(X, U) > t + 1$. Thus the initial state X arrives at the stable state O [cf. (11)] with only one step and we have $E(X) = O$.

(3) The case of $d(X, U) = t + 1$: in view of (18) and (19) we have

$$F_i(X) = \begin{cases} 0 & \text{if } u_i = 0, \\ 1 - x_i & \text{if } u_i = 1. \end{cases} \tag{26}$$

for any pattern $X \in \{0, 1\}^n$ such that $d(X, U) = t + 1$. Thus

$$d(F(X), U) = N^{11}(X, U) \tag{27}$$

in view of (22) and (23)

$$= d(U) - N^{01}(X, U)$$
$$= d(U) - (d(X, U) - N^{10}(X, U))$$
$$= d(U) - (t + 1) + N^{10}(X, U) \geq d(U) - (t + 1)$$

in view of the assumption $t < d(U)/2 - 1$

$$> 2(t + 1) - (t + 1) = t + 1,$$

which means that $F(X)$ satisfies the condition of the case (2). Hence the initial state X arrives at the stable state O [cf. (11)] with only two steps, i.e. $F(F(X)) = O$ and we have $E(X) = O$.

Summing up the above discussion we finally have that $E(X) = U$ if and only if $d(X, U) \leq t$, which completes the proof. □

3. THE BIOLOGICAL DISCUSSION OF SMNNs

Now we consider the SMNN as a biological neural network, then the weight w_{ij} is considered as the efficiency of the synapse from neuron j to neuron i. Firstly, some biological meanings and supposals on the network are given as follows.

As being expressed in the mathematical model, the

We suppose that the learning process is just the modification of synapses. In fact, threshold values of neurons make no contributions to the learning results. We can further suppose that all neurons of the network have the same positive threshold value, i.e., $\theta_1 = \theta_2 = \cdots = \theta_n = \theta > 0$. According to Hebbian learning supposal (Hebb, 1949), we use the following formulae as a Hebbian learning rule for the plastic synapse.

In a period, the state of neuron i is $x_i \in \{0, 1\}$, and the state of neuron j is $x_j \in \{0, 1\}$, then an incremental synapse efficiency (weight) Δw_{ij} is obtained to w_{ij}. Δw_{ij} is computed by

$$\Delta w_{ij} = \begin{cases} +\alpha & \text{if } x_i = x_j = 1, \\ -\alpha & \text{if } x_i \neq x_j, \\ 0 & \text{if } x_i = x_j = 0 \end{cases} \tag{28}$$

where $\alpha > 0$. In fact, this learning rule can be given a unified form as follows:

$$\Delta w_{ij} = \alpha(x_i x_j - \overline{x_i} x_j - x_i \overline{x_j}),$$

where $\overline{x} = 1 - x$.

We suppose that the synapse is in the plastic or stable form, corresponding to short-term or long-term memory, respectively. When the synapse is plastic, the efficiency of it can be modified by the learning rule eqn (28), or the absolute value of it attenuates to zero (forgetting). When the synapse is stable, the efficiency of it $w_{ij} = \pm E (E > 0)$, where E is the saturation value which has two implications: (1) when the absolute value of the plastic synapse efficiency increases up to E, the plastic synapse transforms into the stable one; (2) the stable synapse efficiency $\pm E$ cannot be changed by further learning or forgetting. In the supposal, when $w_{ij} = +E$, the long-term synapse is excitatory; and when $w_{ij} = -E$, the long-term synapse is inhibitory. We further suppose that $E = 1$ for E certainly can be chosen as the unit of weight.

By the above facts and supposals, given a neural network of n neurons with all plastic synapses (W_0, θ), let it learn the simplex pattern $U \in \{0, 1\}^n$, i.e., repeatedly input U to the network and make the network to be in the state of U for a certain time (period). As the learning process continues properly, the synapse efficiencies of the network will come up to the stable values:

$$\begin{cases} 1 & u_i = u_j = 1, i \neq j, \\ [illegible] \end{cases}$$

SMNN of U, with the error-correcting capacity $t = d(U) - \theta - 1$.

So we have shown that the SMNN given by eqns (8) and (9), may be constructed in the brain by repeatedly learning the specified pattern U with the Hebbian learning rule from any network of plastic [illegible]

the brain.

As required, the SMNN certainly has the function of a CAM. We further discuss the two biological functions with the SMNN.

A. The Error Correcting Property

According to Theorem 1, the threshold value θ, error correcting capacity t and $d(U)$ satisfy

$$t = d(U) - \theta - 1. \qquad (30)$$

So when θ is stable (fixed), the error correcting capacity t is directly proportional to $d(U)$. As $d(U)$ increases, the error correcting capacity increases too. In fact, as $d(U)$ [supposing that

$$\frac{d(U)}{2} \leqslant n.$$

excitation (inhibition) on a local field of the brain, this may cause each neuron of the SMNN to be easily excited (inhibited), which means the threshold value is decreased (increased) to a lower (higher) value. According to eqn (30), when the threshold value of the network is decreased, i.e., the excitatory level is [illegible]

that the retrieval of the pattern becomes easy. Alternatively, when the threshold value of the network is increased, i.e., the inhibitory level is increased, the error correcting capacity is decreased so that the retrieval of the pattern becomes difficult. This function seems reasonable.

4. CONCLUSION

We have proposed a model of SMNN to describe the mechanism of pattern memory in the brain. The SMNN is defined to memorize only a simplex pattern with a CAM function. Then a mathematical model of SMNN is constructed for any pattern. Under some physiological supposals (these seem reasonable), the mathematical model can be formed by repeatedly learning the pattern U with Hebbian learning rules from any neural network of plastic synapses. Hence