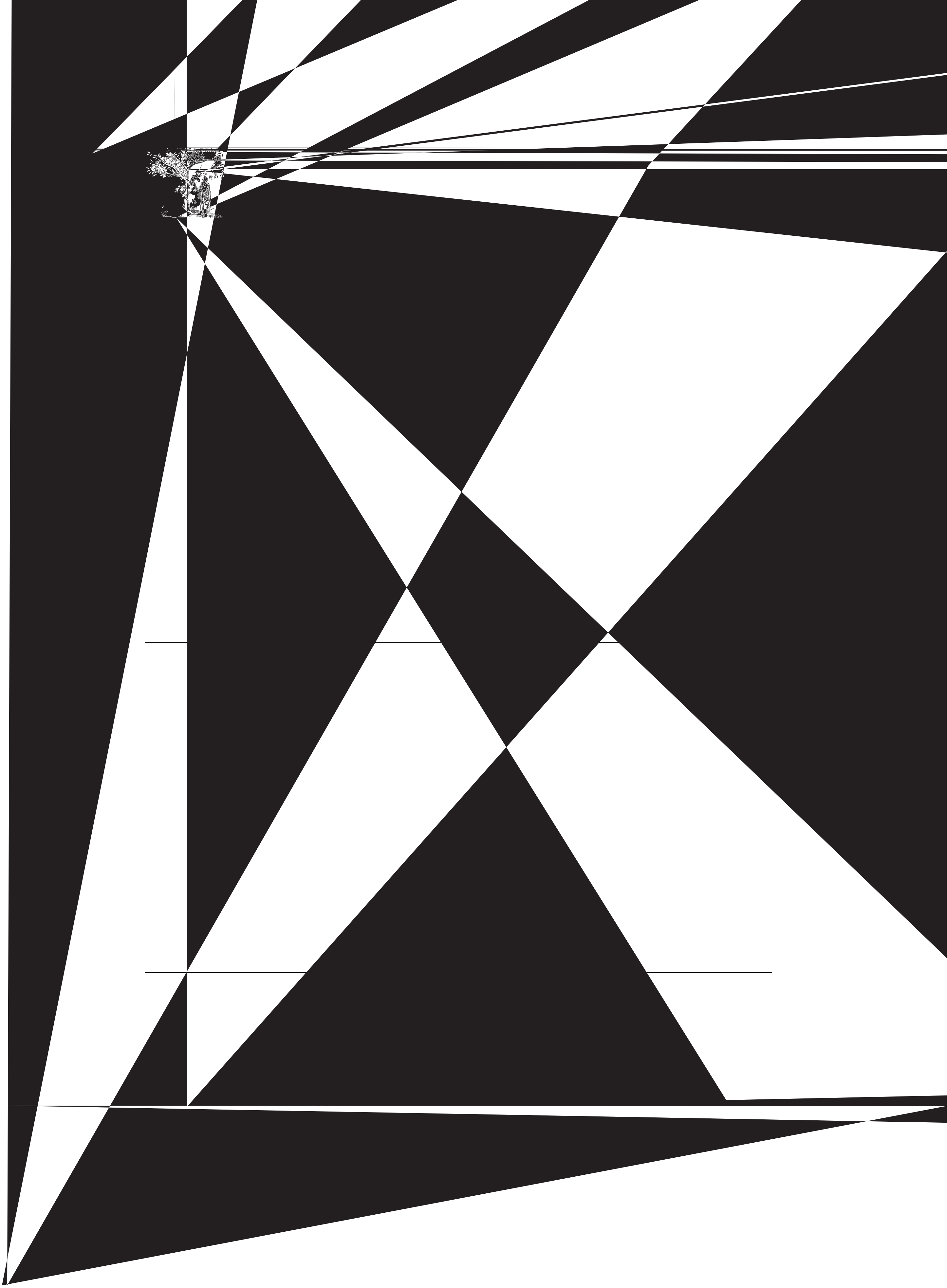

1. Introduction

EM (ML)
(MAP) [3]. EM
(e.g., [2,4,12,15,16,18,19]). G ,
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[9], [5], -N
[6,10], [13] " " [7].
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M [17], [7]
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J [20] H
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H ,
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ML MAP .
I M . [14]
EM
G z. B
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J [20],

z. ,
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z. F
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[14] G .

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G 4 5.

2. The EM algorithm for mixtures of densities from exponential families

2.1. The mixture model

:

$$P(x|\Phi) = \sum_{i=1}^{K} \alpha_i P_i(x|\phi_i), \quad \alpha_i \geqslant 0, \ \sum_{i=1}^{K} \alpha_i = 1, \tag{1}$$

$x = [x_1, \ldots, x_n] \ \in R^n,$ P_i
z $\phi_i \in \Omega_i \subset R^{d_i},$ K
. Φ α_i
$\phi_i,$, $\Phi = (\alpha_1, \ldots, \alpha_K, \phi_1, \ldots, \phi_K) \in \Omega,$

$$\Omega = \left\{ (\alpha_1, \ldots, \alpha_K, \phi_1, \ldots, \phi_K) : \sum_{i=1}^{K} \alpha_i = 1 \quad \alpha_i \geqslant 0, \ \phi_i \in \Omega_i \quad i = 1, \ldots, K \right\}.$$

I $P_i(x|\phi_i) = P_i(x|m_i, \Sigma_i)$ G

$$P_i(x|\phi_i) = P(x|m_i, \Sigma_i) = \frac{1}{(2\pi)^{n/2}(\quad \Sigma_i)^{1/2}} \quad ^{-(1/2)(x-m_i) \ \Sigma_i^{-1}(x-m_i)}, \tag{2}$$

$m_i = [m_{i1}, \ldots, m_{in}]$, $\Sigma_i = (\sigma^i_{kl})_{n \times n}$
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. I , EM
G
[14] . I , EM
. ,
G .
A $q(x|\theta), \theta \in \Theta \subset R^d$ R^n

$$q(x|\theta) = a(\theta)^{-1} b(x) \quad ^{\theta \ t(x)}, \quad x \in R^n, \tag{3}$$

$b(x), t(x)$ x R^n $a(\theta)$

$$a(\theta) = \int_{R^n} b(x) \quad ^{\theta \ t(x)} \ \mu$$

μ R^n. I $b(x) \geqslant 0$
$x \in R^n, a(\theta) < +\infty$ $\theta \in \Theta$ $t(x),$,
. A ,
$b(x).$

O ,

$$P(x|\mathrm{f}) = q(x|\mathrm{y}(\mathrm{f})) = a(\mathrm{f})^{-1}b(x) \quad \mathrm{y}(\mathrm{f}) \quad t(x), \quad x \in R^n$$

'' '' $\mathrm{f} = E_{\mathrm{y}}(t(X))$

([1] [18]).

$P(x|\mathrm{f})$ m S,

$U(x|\mathrm{f})$:

$$P(x|\mathrm{f}) \leqslant U(x|\mathrm{f}) = w(x)(1 \quad)^{-c_1} \quad -\mathrm{r}(1/\mathrm{Z}(x))^{-c_2}, \tag{4}$$

$$\mathrm{Z}(x) = \frac{(1 \quad)^{\mathrm{n}}}{\|x-m\|}$$

l S $P(x|\mathrm{f})$. M , c_1, c_2, r n

, $w(x)$ $x_1, \ldots, x_n$

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G .

I , $P_i(x|\mathrm{f}_i)$ '' '' z -

$\mathrm{f}_i \in \mathrm{O}_i \subset R^{d_i}$:

$$P_i(x|\mathrm{f}_i) = a_i(\mathrm{f}_i)^{-1}b_i(x) \quad \mathrm{y}_i(\mathrm{f}_i) \quad t_i(x), \quad x \in R^n \tag{5}$$

$\mathrm{F}^* = (\mathrm{a}_1^*, \ldots, \mathrm{a}_K^*, \mathrm{f}_1^*, \ldots, \mathrm{f}_K^*)$ '' ''

, ,

F^*. A , $t_i(x)$

$x_1, \ldots, x_n$.

M , $P_i(x|\mathrm{f}_i^*)$:

$$P_i(x|\mathrm{f}_i^*) \leqslant U_i(x|\mathrm{f}_i^*) = w(x)(1^i \quad)^{-c_1} \quad -\mathrm{r}(1/\mathrm{Z}_i(x))^{c_2}, \tag{6}$$

$$\mathrm{Z}_i(x) = \frac{(1^i \quad)^{\mathrm{n}_i}}{\|x-m_i^*\|}$$

m_i^* 1^i

S_i^* $P_i(x|\mathrm{f}_i^*)$, . c_1, c_2, r $w(x)$ i .

F , 1^i . n

n_i r $U_i(x|\mathrm{f}_i^*)$

. A ,

$$\mathrm{Z}_i(x) = \frac{(1^i \quad)^{\mathrm{n}}}{\|x-m_i^*\|}, \quad i = 1, \ldots, K,$$

n .

2.2. The EM algorithm and its asymptotic convergence rate

$\mathscr{S}_N = \{x^{(t)} : t = 1, \ldots, N\}$

$f_1, \ldots, f_K$. I F $= (a_1, \ldots, a_K, f_1, \ldots, f_K)$

ML - $L(F) = \sum_{t=1}^{N}$ $P(x^{(t)}|F)$

E . (1), EM

$$a_i^+ = \frac{1}{N}\sum_{t=1}^{N} \frac{a_i P_i(x^{(t)}| f_i)}{P(x^{(t)}|F\ )}, \tag{7}$$

$$f_i^+ = \left\{\sum_{t=1}^{N} t_i(x^{(t)}) \frac{a_i P_i(x^{(t)}| f_i)}{P(x^{(t)}|F\ )}\right\} \Bigg/ \left\{\sum_{t=1}^{N} \frac{a_i P_i(x^{(t)}| f_i)}{P(x^{(t)}|F\ )}\right\}, \tag{8}$$

$i = 1, \ldots, K.$

$L(F)$ [3,19] . M ,

, EM

z $L(F)$ [18] . I ,

EM F^* (. .,

z N $\mathscr{S}_N$, EM F^N

$_{N\to\infty} F^N = F^*$), *z*

F^*.

I [18] , EM

$F^+ = G(F\ )$

$$F^+ - F^N = G(F\ ) - G(F^N) = G'(F^N)(F\ - F^N) + O(\|F\ - F^N\|^2) \tag{9}$$

F O F^N, $G'(F)$ J $G(F)$ F^N $O(x)$

$x \to 0$. B ,

N , , $G'(F^N)$

$E(G'(F^*)) = I - Q(F^*)R(F^*),$

$$Q(F^*) = diag(a_1^*, \ldots, a_K^*, a_1^{*-1} P_1, \ldots, a_K^{*-1} P_K) \tag{10}$$

$$P_i = \int_{R^n} [t_i(x) - f_i^*][t_i(x) - f_i^*]\ \ P_i(x| f_i^*)\ \ m$$

$$R(F^*) = \int_{R^n} V(x)V(x)\ \ P(x|F^*)\ \ m \tag{11}$$

$$V(x) = (b_1(x), \ldots, b_K(x), a_1^* b_1(x) G_1(x)\ \ , \ldots, a_K^* b_K(x) G_K(x)\ \ )\ \ ,$$

$$b_i(x) = P_i(x| f_i^*)/P(x|F^*),$$

$$G_i(x) = P_i^{-1}[t_i(x) - f_i^*].$$

H , $E(\cdot) = E_{\Phi^*}(\cdot)$. I E . (9) EM Φ^N $\|G'(\Phi^N)\|$. B N , r EM Φ^*:

$$
\begin{aligned}
r &\leqslant {}_{N\to\infty} \|G'(\Phi^N)\| = \left\| {}_{N\to\infty} G'(\Phi^N) \right\| \\
&= \|E(G'(\Phi^*))\| = \|I - Q(\Phi^*)R(\Phi^*)\|.
\end{aligned}
\tag{12}
$$

I , EM

z z .

3. The main result

3.1. The measures of the overlap

[14

G . E . (1) Φ^*:

$$h_i(x) = \frac{\alpha_i^* P_i(x|\phi_i^*)}{\sum_{j=1}^{K} \alpha_j^* P_j(x|\phi_j^*)} \qquad i = 1, \ldots, K. \tag{13}$$

I E . (11)

$$h_i(x) = \alpha_i^* \beta_i(x). \tag{14}$$

$$\gamma_{ij}(x) = (\delta_{ij} - h_i(x))h_j(x) \qquad i,j = 1, \ldots, K, \tag{15}$$

δ_{ij} K . , :

$$e_{ij}(\Phi^*) = \int_{R^n} |\gamma_{ij}(x)| P(x|\Phi^*)\, \mathrm{d}\mu$$

$i,j = 1, 2, \ldots, K,$ $e_{ij}(\Phi^*) \leqslant 1$ $|\gamma_{ij}(x)| \leqslant 1.$ F $i \neq j$, $e_{ij}(\Phi^*)$ i j . $P_i(x|\phi_i^*)$ $P_j(x|\phi_j^*)$ x, $h_i(x)h_j(x)$

$e(\mathrm{F}^*) = 0$

$h_i(x)h_j(x) = 0$ $i \neq j$

, . ., .

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z EM N

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$e(\mathrm{F}^*)$ *z* . A

EM

z ,

$e(\mathrm{F}^*)$.

3.2. Regular conditions and lemmas

:

(1) *Nondegenerate condition on the mixing proportions*:

:

$$\mathrm{a}_i^* \geqslant \mathrm{a} \qquad i = 1, \ldots, K, \tag{16}$$

a . I *z* ,

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(2) *Uniform attenuating condition on the eigenvalues of the covariance matrices*:

S_i^* i $\mathrm{l}_{i1}, \ldots, \mathrm{l}_{in}$

.

$$\mathrm{b}\,\mathrm{l}(\mathrm{F}^*) \leqslant \mathrm{l}_{ij} \leqslant \mathrm{l}(\mathrm{F}^*) \qquad i = 1, \ldots, K,\ k = 1, \ldots, n, \tag{17}$$

b $\mathrm{l}(\mathrm{F}^*)$

$\mathrm{S}_1^*, \ldots, \mathrm{S}_K^*$, . .,

$$\mathrm{l}(\mathrm{F}^*) = \quad _{i,j} \quad \mathrm{l}_{ij}$$

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z *z* . I

E . (17) K

, . .,

$$1 \leqslant \mathrm{k}(\mathrm{S}_i^*) \leqslant B' \qquad i = 1, \ldots, K,$$

$\mathrm{k}(\mathrm{S}_i^*)$ S_i^* B' .

(3) *Regular condition on the mean vectors*:

, . ., $m_1^*, \ldots, m_K^*$,

$$\mathrm{m}D \quad (\mathrm{F}^*) \leqslant D \quad (\mathrm{F}^*) \leqslant \|m_i^* - m_j^*\| \leqslant D \quad (\mathrm{F}^*) \qquad i \neq j, \tag{18}$$

$D \quad (\mathrm{F}^*) = \quad {}_{i \neq j}\|m_i^* - m_j^*\|$, $D \quad (\mathrm{F}^*) = \quad {}_{i \neq j}\|m_i^* - m_j^*\|$, m

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. M ,

z , m_i^*, m_j^* , . .,

T $\|m_i^* - m_j^*\| \geqslant T$ $i \neq j$. I ,

E . (18) $m_1^*, \ldots, m_K^*$.

$\|\mathrm{P}_i^{-1}\|$, P_i E . (10).

$P_i(x|\,\mathrm{f}_i^*)$,

[11 :

$\mathrm{P}_i \not\asymp$

$\mathrm{Z}(\mathrm{F}^*) \to 0$. , $\mathrm{Z}(\mathrm{F}^*) \to 0$ $e(\mathrm{F}^*) \to 0$ $\mathrm{Z}(\mathrm{F}^*) \to 0$ $\mathrm{Z}(\mathrm{F}^*) \to 0$.
A
, $e(\mathrm{F}^*)$
z , $\mathrm{Z}(\mathrm{F}^*) \to 0$ $e(\mathrm{F}^*) \to 0$.
$\mathrm{Z}(\mathrm{F}^*)$, . . ., F^*
$\mathrm{Z}(\mathrm{F}^*)$,

$$f(\mathrm{Z}) = \ _{\mathrm{Z}(\mathrm{F}^*)=\mathrm{Z}} e(\mathrm{F}^*) \tag{21}$$

$e(\mathrm{F}^*)$ 1. B ,

$$e_{ij}(\mathrm{F}^*) \leqslant e(\mathrm{F}^*) \leqslant f(\mathrm{Z}(\mathrm{F}^*)) \qquad i \neq j. \tag{22}$$

F , (A A).

Lemma 1. *Suppose that a mixture of K densities from the bell sheltered exponential families of the parameter* F^* *satisfies Conditions* (1) (3). *As* $\mathrm{Z}(\mathrm{F}^*)$ *tends to zero*, *we have*

( ) $\mathrm{Z}(\mathrm{F}^*), \mathrm{Z}_i(m_j^*)$ *and* $\mathrm{Z}_j(m_i^*)$ *are the equivalent infinitesimals.*
( ) *For* $i \neq j$, *we have*

$$\|m_i^*\| \leqslant T' \|m_i^* - m_j^*\|, \tag{23}$$

where T' *is a positive number.*
( ) *For any two nonnegative numbers with* $p + q > 0$, *we have*

$$\|m_i^* - m_j^*\|^p (1^i \quad)^{-\mathrm{n}q} \leqslant O(\mathrm{Z}^{-p \vee q}(\mathrm{F}^*)), \tag{24}$$

where $p \vee q = \quad \{p, q\}$.

Lemma 2. *Suppose that a mixture of K densities from the bell sheltered exponential families of the parameter* F^* *satisfies Conditions* (1) (3). *As* $\mathrm{Z}(\mathrm{F}^*)$ *tends to zero*, *we have for each i*

( ) $$\|\mathrm{P}_i\| \leqslant \mathrm{c}\, \|m_i^* - m_j^*\|^p, \tag{25}$$

where $j \neq i$, c *and* p *are some positive numbers.*

( ) $$E(\|t_i(X) - \mathrm{f}_i^*\|^2) \leqslant \mathrm{u} M_i^q(\mathrm{F}^*), \tag{26}$$

where $M_i(\mathrm{F}^*) = \quad _{j \neq i} \|m_i^* - m_j^*\|$, u *and* q *are some positive numbers.*

Lemma 3. *Suppose that a mixture of K densities from the bell sheltered exponential families of the parameter* F^* *satisfies Conditions* (1) (3) *and* $\mathrm{Z}(\mathrm{F}^*) \to 0$ *as an infinitesimal*, *we have*

$$f^{\mathrm{e}}(\mathrm{Z}(\mathrm{F}^*)) = o(\mathrm{Z}^p(\mathrm{F}^*)), \tag{27}$$

where $\varepsilon > 0$, *p is any positive number and* $o(x)$ *means that it is a higher order infinitesimal as* $x \to 0$.

. E ,
$e(\Phi^*)$ $Z(\Phi^*)$ L 3
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3.3. The main theorem

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Theorem 1. *Given a mixture of K densities from the bell sheltered exponential families of the parameter* Φ^* *that satisfies Conditions* (1) (4), *as* $e(\Phi^*)$ *tends to zero as an infinitesimal, we have*

$$r \leqslant \|E(G'(\Phi^*))\| = o(\ ^{0.5-\varepsilon}(\Phi^*)), \tag{28}$$

where ε *is an arbitrarily small positive number.*

A , ,
, , $e(\Phi^*) \to 0$, $\|E(G'(\Phi^*))\|$
$^{0.5-\varepsilon}(\Phi^*)$. , (Φ^*) ,
EM Φ^*
$^{0.5-\varepsilon}(\Phi^*)$. , $e(\Phi^*)$ N ,
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“
[16] P -EM [13] , EM
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Proof of Theorem 1. $Q(\mathrm{F}^*)R(\mathrm{F}^*)$.
A $Q(\mathrm{F}^*)$ $R(\mathrm{F}^*)$,
$Q(\mathrm{F}^*)R(\mathrm{F}^*)$, :

$$Q(\mathrm{F}^*)R(\mathrm{F}^*) = diag[diag[\mathscr{A}\ ], \mathrm{a}^{*-1}\,\mathrm{P}_1, \ldots, \mathrm{a}_K^{*-1}\,\mathrm{P}_K] \times \begin{pmatrix} R_{\mathrm{b,b}} & R_{\mathrm{b,G_1}} & \cdots & R_{\mathrm{b,G}_K} \\ R_{\mathrm{G_1,b}} & R_{\mathrm{G_1,G_1}} & \cdots & R_{\mathrm{G_1,G}_K} \\ \vdots & \vdots & \ddots & \vdots \\ R_{\mathrm{G}_K\mathrm{,b}} & R_{\mathrm{G}_K\mathrm{,G_1}} & \cdots & R_{\mathrm{G}_K\mathrm{,G}_K} \end{pmatrix}$$

$$= \begin{pmatrix} diag[\mathscr{A}\ ]R_{\mathrm{b,b}} & diag[\mathscr{A}\ ]R_{\mathrm{b,G_1}} & \cdots & diag[\mathscr{A}\ ]R_{\mathrm{b,G}_K} \\ \mathrm{a}_1^{*-1}\,\mathrm{P}_1 R_{\mathrm{G_1,b}} & \mathrm{a}_1^{*-1}\,\mathrm{P}_1 R_{\mathrm{G_1,G_1}} & \cdots & \mathrm{a}_1^{*-1}\,\mathrm{P}_1 R_{\mathrm{G_1,G}_K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{a}_K^{*-1}\,\mathrm{P}_K R_{\mathrm{G}_K\mathrm{,b}} & \mathrm{a}_K^{*-1}\,\mathrm{P}_K R_{\mathrm{G}_K\mathrm{,G_1}} & \cdots & \mathrm{a}_K^{*-1}\,\mathrm{P}_K R_{\mathrm{G}_K\mathrm{,G}_K} \end{pmatrix},$$

$\mathrm{b}(x) = [\mathrm{b}_1(x), \ldots, \mathrm{b}_K(x)]$ $\mathscr{A} = [\mathrm{a}_1^*, \ldots, \mathrm{a}_K^*]$.
$R(\mathrm{F}^*)$ $V(x)$

$$V(x) = [\mathrm{b}(x)\ , \mathrm{a}_1^*\mathrm{b}_1(x)\mathrm{G}_1(x)\ , \ldots, \mathrm{a}_K^*\mathrm{b}_K(x)\mathrm{G}_K(x)\ ]\ .$$

( ) *The computation of* $diag[\mathscr{A}\ ]R_{\mathrm{b,b}}$: F $\mathrm{b}_i(x)$
$h_i(x) = \mathrm{a}_i^*\mathrm{b}_i(x)$,

$$\int_{R^n} \mathrm{b}_i(x)\mathrm{b}_j(x)P(x|\mathrm{F}^*)\ \ \mathrm{m} = \frac{1}{\mathrm{a}_i^*\mathrm{a}_j^*} e_{ij}(\mathrm{F}^*) \qquad i \neq j,$$

$$\int_{R^n} \mathrm{b}_i^2(x)P(x|\mathrm{F}^*)\ \ \mathrm{m} = \frac{1}{\mathrm{a}_i^*} - \frac{1}{(\mathrm{a}_i^*)^2} e_{ii}(\mathrm{F}^*)$$

$$diag[\mathscr{A}\ ]R_{\mathrm{b,b}}\ = I_K + \begin{pmatrix} -\mathrm{a}_1^{*-1}e_{11}(\mathrm{F}^*) & \mathrm{a}_2^{*-1}e_{12}(\mathrm{F}^*) & \cdots & \mathrm{a}_K^{*-1}e_{1K}(\mathrm{F}^*) \\ \mathrm{a}_1^{*-1}e_{21}(\mathrm{F}^*) & -\mathrm{a}_2^{*-1}e_{22}(\mathrm{F}^*) & \cdots & \mathrm{a}_K^{*-1}e_{2K}(\mathrm{F}^*) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{a}_1^{*-1}e_{K1}(\mathrm{F}^*) & \mathrm{a}_2^{*-1}e_{K2}(\mathrm{F}^*) & \cdots & -\mathrm{a}_K^{*-1}e_{KK}(\mathrm{F}^*) \end{pmatrix}.$$

B

$$\frac{1}{\mathrm{a}_j^*} e_{ij}(\mathrm{F}^*) \leqslant \frac{1}{\mathrm{a}} e_{ij}(\mathrm{F}^*) = o(\ \ ^{0.5-\mathrm{e}}(\mathrm{F}^*)),$$

$$diag[\mathscr{A}\ \ ]R_{\mathrm{b,b}} \ = I_K + o$$

, . I

C z $|g_{ij}(x)| \leqslant 1$

$$
\begin{aligned}
&|E(h_j(X)(h_i(X) - d_{ij})(t_{i,k}(X) - f^*_{i,k}))| \\
&\quad \leqslant E(|h_j(X)(h_i(X) - d_{ij})||(t_{i,k}(X) - f^*_{i,k})|) \\
&\quad \leqslant E^{1/2}(g^2_{ij}(X))E^{1/2}((t_{i,k}(X) - f^*_{i,k})^2) \\
&\quad \leqslant E^{1/2}(|g_{ij}(X))E^{1/2}((t_{i,k}(X) - f^*_{i,k})^2) \\
&\quad \leqslant \sqrt{e_{ij}(F^*)}E^{1/2}((t_{i,k}(X) - f^*_{i,k})^2).
\end{aligned}
$$

A L 2, $E(\|t_i(X) - f^*_i\|^2|F^*)$ $uM^q_i(F^*)$. , $E^{1/2}((t_{i,k}(X) - f^*_{i,k})^2)$ $\sqrt{uM^q_i(F^*)}$.

,

$$E(diag[\mathscr{A}\ ]a^*_i b_i(X)b(X)(t_i(X) - f^*_i)\ ) = O(M^{q/2}_i(F^*)e^{0.5}(F^*)).$$

A L 1 3, $M^{q/2}_i(F^*)\ ^{0.5}(F^*)$ $e(F^*)$ $Z(F^*)$ z . I

$$\|E(diag[\mathscr{A}\ ]a^*_i b_i(X)b(X)(t_i(X) - f^*_i)\ )\| = O(M^{q/2}_i(F^*)\ ^{0.5}(F^*)).$$

M ,

$$\|diag[\mathscr{A}\ ]R_{b,G_i}\| \leqslant \|E(diag[\mathscr{A}\ ]a^*_i b_i(X)b(X)(t_i(X) - f^*_i)\ )\|\|P^{-1}_i\|$$

$$\|P^{-1}_i\| = \|I(f^*_i)\| \leqslant O(\|m^*_i\|^{t_1}(1^i\quad)^{-t_2})$$

C (4). ,

$$\|diag[\mathscr{A}\ ]R_{b,G_i}\| \leqslant u\|m^*_i - m^*_{j'}\|^{q_1}(1^i\quad)^{-q_2}\ ^{0.5}(F^*),$$

$q_1 = (q/2) + t_1$, $q_2 = t_2$, u . , L 1 3

$$\|diag[\mathscr{A}\ ]R_{b,G_i}\| \leqslant O(Z^{-q_1 \vee q_2}(F^*))e^{0.5}(F^*) = o(e^{0.5-e}(F^*)).$$

B ,

$$diag[\mathscr{A}\ ]R_{b,G_i} = o(\ ^{0.5-e}(F^*)).$$

( ) *The computation of* $a^{*-1}_i P_i R_{G_i,b}$ $(i = 1, \ldots, K)$: A (1) L 2, $a^{*-1}_i\|P_i\|$ $(1/a)c\|m^*_i - m^*_j\|^p$, $j \neq i$, c p . B $R_{G_i,b} = R_{b,G_i}$, ( ) :

$$a^{*-1}_i P_i R_{G_i,b} = o(\ ^{0.5-e}(F^*)).$$

( ) *The computation of* $\alpha_i^{*-1}\Pi_i R_{\Gamma_i,\Gamma_i}$ $(i=1,\ldots,K)$: B $V(x)$,

$$\alpha_i^{*-1}\Pi_i R_{\Gamma_i,\Gamma_i} = \alpha_i^{*-1}\Pi_i E(h_i^2(X)\Gamma_i(X)\Gamma_i(X))$$
$$= \alpha_i^{*-1}E(h_i^2(X)(t_i(X)-\phi_i^*)(t_i(X)-\phi_i^*))\Pi_i^{-1}$$
$$= I_{d_i} + \alpha_i^{*-1}E(h_i(X)(h_i(X)-1)(t_i(X)-\phi_i^*)(t_i(X)-\phi_i^*))\Pi_i^{-1},$$

:

$$\Pi_i E(h_i(X)\Gamma_i(X)\Gamma_i(X)) = \alpha_i^* I_{d_i}.$$

F , $E(\|t_i(X)-\phi_i^*\|^2|\Phi^*)$ $uM_i^q(\Phi^*)$ α_i^{*-1} , , :

$$\alpha_i^{*-1}E(h_i(X)(h_i(X)-1)(t_i(X)-\phi_i^*)(t_i(X)-\phi_i^*))\Pi_i^{-1} = o(\ ^{0.5-\varepsilon}(\Phi^*))$$

,

$$\alpha_i^{*-1}\Pi_i R_{\Gamma_i,\Gamma_i} = I_{d_i} + o(\ ^{0.5-\varepsilon}(\Phi^*)).$$

( ) *The computation of* $\alpha_i^{*-1}\Pi_i R_{\Gamma_i,\Gamma_j}$ $(j\neq i)$: B $V(x)$,

$$\alpha_i^{*-1}\Pi_i R_{\Gamma_i,\Gamma_j} = \alpha_i^{*-1}E(\alpha_i^*\beta_i(X)\alpha_j^*\beta_j(X)(t_i(X)-\phi_i^*)(t_j(X)-\phi_j^*))\Pi_j^{-1}$$
$$= \alpha_i^{*-1}E(h_i(X)h_j(X)(t_i(X)-\phi_i^*)(t_j(X)-\phi_j^*))\Pi_j^{-1}.$$

( ),

$$\alpha_i^{*-1}\Pi_i R_{\Gamma_i,\Gamma_j} = o(\ ^{0.5-\varepsilon}(\Phi^*)).$$

( ) ( ), :

$$Q(\Phi^*)R(\Phi^*) = I + o(\ ^{0.5-\varepsilon}).$$

, E . (12),

$$r \leqslant \|I - Q(\Phi^*)R(\Phi^*)\| = o(\ ^{0.5-\varepsilon}(\Phi^*)). \quad \square$$

4. A typical class: Gaussian mixtures

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G

. A [1] , G $P_i(x|m_i,\Sigma_i)$ E . (2) $\psi_i = (\Sigma_i^{-1}m_i, \Sigma_i^{-1})$

$t_i(x) = (x, -\frac{1}{2}xx\ )$. , ϕ_i, ψ_i, $(m_i, -\frac{1}{2}(\Sigma_i + m_im_i\ ))$ (m_i, Σ_i)

.

N , G

-

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Lemma 4. *Suppose that* $P_i(x|\,\mathrm{f}_i^*) = P_i(x|m_i^*, \mathrm{S}_i^*)$ *is a Gaussian distribution with the mean* m_i^* *and the covariance matrix* S_i^*, *and that the condition number of* S_i^*, *i.e.*, $\mathrm{k}(\mathrm{S}_i^*)$, *is upper bounded by* B'. *We have that* $P_i(x|\,\hat{\mathrm{f}}_i^*)$ *is bell-sheltered, i.e.*,

$$P_i(x|\,\mathrm{f}_i^*) = P_i(x|m_i^*, \mathrm{S}_i^*) \leqslant b\frac{1}{(1^i \quad)^{n/2}} \quad ^{-(1/21^i \quad)\|x-m_i^*\|^2}, \tag{29}$$

where b is a positive number.

Proof. B $y = U_i(x - m_i^*)$

$$P(y|1^i \quad) = \frac{1}{(2\mathrm{p}1^i \quad)^{n/2}} \quad ^{-(1/21^i \quad)\|y\|^2},$$

$$P_i(x|m_i^*, \mathrm{S}_i^*) \leqslant B'^{n/2} P(y|1^i \quad),$$

$\mathrm{k}(\mathrm{S}_i^*) \leqslant B'$. M , $\|y\| = \|x - m_i^*\|$,

$$P_i(x|m_i^*, \mathrm{S}_i^*) \leqslant b\frac{1}{(1^i \quad)^{n/2}} \quad ^{-(1/21^i \quad)\|x-m_i^*\|^2},$$

$b = (B'/2\mathrm{p})^{n/2}$. □

B L 4, (1) (3), G F^*

K .

M , $P_i(x|\,\mathrm{f}_i^*) = P_i(x|m_i^*, \mathrm{S}_i^*)$, $t_i(x)$

:

$$t_i(x) = \begin{cases} x & m_i^*, \\ -\frac{1}{2}xx & -\frac{1}{2}(\mathrm{S}_i^* + m_i^*(m_i^*) \quad). \end{cases}$$

$t_i(x)$ $x_1, \ldots, x_n$.

, G F^* (1) (3)

(4).

F , G , (4) (2),

. F ,

$\mathrm{f}_i^* = [(m_i^*) \quad, vec[\dot{\mathrm{S}}_i^*] \quad]$, $\dot{\mathrm{S}}_i^* = -\frac{1}{2}(\mathrm{S}_i^* + m_i^*(m_i^*) \quad)$, $\hat{\mathrm{f}}_i^* = [(m_i^*) \quad, vec[\mathrm{S}_i^*] \quad]$.

Lemma 5. *Suppose that* $P_i(x|\,\mathrm{f}_i^*) = P_i(x|m_i^*, \mathrm{S}_i^*)$ *is a Gaussian density and* $\mathrm{k}(\mathrm{S}_i^*)$ *is upper bounded. As* 1^i *tends to zero, we have*

$$\|I(\mathrm{f}_i^*)\| = O((1^i \quad)^{-\mathrm{t}}), \tag{30}$$

where t *is a positive number.*

Proof. B ,

$$\frac{\partial P_i(x|m_i^*, \mathrm{S}_i^*)}{\partial m_i^*} = (x - m_i^*)\mathrm{S}_i^* P_i(x|m_i^*, \mathrm{S}_i^*), \tag{31}$$

$$\frac{\partial P_i(x|m_i^*, \mathrm{S}_i^*)}{\partial \mathrm{S}_i^*} = -\frac{1}{2}(\mathrm{S}_i^{*-1} - \mathrm{S}_i^{*-1}(x - m_i^*)(x - m_i^*)\ \ \mathrm{S}_i^{*-1})P_i(x|m_i^*, \mathrm{S}_i^*). \tag{32}$$

A F ,

$$\begin{aligned} I(\mathrm{f}_i^*) &= E_{\mathrm{f}_i^*}\left(\left(\frac{\partial P_i(X|\,\mathrm{f}_i^*)}{\partial\,\mathrm{f}_i^*}\right)\left(\frac{\partial P_i(X|\,\mathrm{f}_i^*)}{\partial\,\mathrm{f}_i^*}\right)\ \ \right) \\ &= E_{\mathrm{f}_i^*}\left(\frac{\partial(\hat{\mathrm{f}}_i^*)}{\partial\,\mathrm{f}_i^*}\ \left(\frac{\partial P_i(X|\,\mathrm{f}_i^*)}{\partial\,\hat{\mathrm{f}}_i^*}\right)\left(\frac{\partial P_i(X|\,\mathrm{f}_i^*)}{\partial\,\hat{\mathrm{f}}_i^*}\right)\ \ \left(\frac{\partial(\hat{\mathrm{f}}_i^*)}{\partial\,\mathrm{f}_i^*}\right)\ \ \right) \\ &= \frac{\partial(\hat{\mathrm{f}}_i^*)}{\partial\,\mathrm{f}_i^*} I(\hat{\mathrm{f}}_i^*)\left(\frac{\partial(\hat{\mathrm{f}}_i^*)}{\partial\,\mathrm{f}_i^*}\right)\ , \end{aligned}$$

$$I(\hat{\mathrm{f}}_i^*) = E_{\mathrm{f}_i^*}\left(\left(\frac{\partial P_i(X|\,\mathrm{f}_i^*)}{\partial\,\hat{\mathrm{f}}_i^*}\right)\left(\frac{\partial P_i(X|\,\mathrm{f}_i^*)}{\partial\,\hat{\mathrm{f}}_i^*}\right)\ \ \right).$$

I E . (31) (32)
$P_i^3(x|m_i^*, \mathrm{S}_i^*)$ $I(\hat{\mathrm{f}}_i^*)$ G
$P_i(x|m_i^*, \frac{1}{3}\mathrm{S}_i^*)$ $|\mathrm{S}_i^*|$
,

$$I(\hat{\mathrm{f}}_i^*) = E_{(m_i^*,(1/3)\mathrm{S}_i^*)}(G(X, \mathrm{f}_i^*)),$$

$G(x, \mathrm{f}_i^*)$ $x - m_i^*$ S_i^*.
$y = x - m_i^*$,

$$I(\hat{\mathrm{f}}_i^*) = E_{(0,(1/3)\mathrm{S}_i^*)}(G(Y, \mathrm{S}_i^*)),$$

$G(y, \mathrm{S}_i^*)$ $g_{pq}(y, \mathrm{S}_i^*)$
$y_1, \ldots, y_n$. I S_i^{*-1}

$$\mathrm{S}_i^{*-1} = |\mathrm{S}_i^*|^{-1}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d_i} \\ a_{21} & a_{22} & \cdots & a_{2d_i} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d_i1} & a_{d_i2} & \cdots & a_{d_id_i} \end{pmatrix},$$

a_{kl} s_{kl}^{*i} S_i^*,
$g_{pq}(y, \mathrm{S}_i^*)$ s_{kl}^{*j}
$|\mathrm{S}_i^*|$. s_{kl}^{*i}

, t
$(1^i\quad)^t g_{pq}(y, \mathrm{S}_i^*)$. B
G l $<B$, $E_{(0,1/3\mathrm{S}_i^*)}((1^i\quad)^t g_{pq}(Y, \mathrm{S}_i^*))$
. ,

$$
\begin{aligned}
\|I(\hat{\mathrm{f}}_i^*)\| &= \|(1^i\quad)^{-t}(1^i\quad)^t I(\mathrm{f}_i^*)\| \\
&= (1^i\quad)^{-t}\|(1^i\quad)^t I(\mathrm{f}_i^*)\| \\
&\leqslant \mathrm{o}(1^i\quad)^{-t},
\end{aligned}
$$

o . B

$$
\|I(\mathrm{f}_i^*)\| \leqslant \left\|\frac{\partial(\hat{\mathrm{f}}_i^*)}{\partial \mathrm{f}_i^*}\right\| \|I(\hat{\mathrm{f}}_i^*)\| \left\| \left(\frac{\partial(\hat{\mathrm{f}}_i^*)}{\partial \mathrm{f}_i^*}\right) \quad \right\| = 4\|I(\hat{\mathrm{f}}_i^*)\|,
$$

$\|\partial(\hat{\mathrm{f}}_i^*)\ /\partial \mathrm{f}_i^*\| = \|(\partial(\hat{\mathrm{f}}_i^*)\ /\partial \mathrm{f}_i^*)\ \| = 2$,
, E . (30) . □

, (1) (3)
EM G
, ,

Theorem 2. *Given a Gaussian mixture of K densities of the parameter* F^* *that satisfies conditions* (1) (3), *as* $e(\mathrm{F}^*)$ *tends to zero as an infinitesimal*, *we have*

$$
\|G'(\mathrm{F}^*)\| = o(\ ^{0.5-\mathrm{e}}(\mathrm{F}^*)), \tag{33}
$$

where e *is an arbitrarily small positive number*.

I , 1 G
(1) (3) .

5. Conclusions

I ,
, EM
-N . M ,
z .

Acknowledgements

G G C
H K A (P . C HK4225/04E) G N
F C (P . 60071004).

Appendix

Proof of Lemma 1. ( ). F ,
$Z(F^*) = \quad {}_{i \neq j} Z_i(m_j^*) = Z_{i'}(m_{j'}^*)$. A (2) (3),
$a_1, a_2, b_1, b_2, c_1, c_2$

$$a_1(1^{i'} \quad)^n \leqslant (1^i \quad)^n \leqslant a_2(1^{i'} \quad)^n, \tag{34}$$

$$b_1(1^i \quad)^n \leqslant (1^j \quad)^n \leqslant b_2(1^i \quad)^n, \tag{35}$$

$$c_1\|m_{i'}^* - m_{j'}^*\| \leqslant \|m_i^* - m_j^*\| \leqslant c_2\|m_{i'}^* - m_{j'}^*\|. \tag{36}$$

C E . (34) (35) E . (36),
a'_1, a'_2, b'_1, b'_2

$$a'_1 Z(F^*) \leqslant Z_i(m_j^*) \leqslant a'_2 Z(F^*),$$

$$b'_1 Z_i(m_j^*) \leqslant Z_j(m_i^*) \leqslant b'_2 Z_i(m_j^*).$$

, $Z(F^*), Z_i(m_j^*)$ $Z_j(m_i^*)$.
( ), $\|m_i^* - m_j^*\|$ $Z(F^*) \to 0$, E . (23)
. I $\|m_i^* - m_j^*\|$ $Z(F^*) \to 0$,
$\|m_i^*\| \leqslant \|m_i^* - m_j^*\| + \|m_j^*\|$, E . (23) $\|m_i^*\|$ $\|m_j^*\|$
. O , $\|m_i^*\|$ $\|m_j^*\|$,
$\|m_i^*\|$ $\|m_i^* - m_j^*\|$,
E . (23). , ( ) .
F , ( ) .
I $p = q > 0$, ( ),

$$\|m_i^* - m_j^*\|^p(1^i \quad)^{-nq} = \|m_i^* - m_j^*\|^p(1^i \quad)^{-np}$$
$$= (Z_i(m_j^*))^{-p} = O(Z^{-p}(F^*)) = O(Z^{-p \vee q}(F^*)).$$

I $p > q$, 1^i ( ),

$$\|m_i^* - m_j^*\|^p(1^i \quad)^{-nq} \leqslant O(Z^{-p}(F^*)) = O(Z^{-p \vee q}(F^*)).$$

I $p < q$, $\|m_i^* - m_j^*\| \geqslant T$,

$$\|m_i^* - m_j^*\|^p(1^i \quad)^{-nq} \leqslant O(Z^{-q}(F^*)) = O(Z^{-p \vee q}(F^*)).$$

,

$$\|m_i^* - m_j^*\|^p(1^i \quad)^{-nq} \leqslant O(Z^{-p \vee q}(F^*)). \qquad \square$$

Proof of Lemma 2. ( ). A ,

$$\|P_i\| = \|E_{f_i^*}((t_i(X) - f_i^*)(t_i(X) - f_i^*) \quad)\| \leqslant E_{f_i^*}(\|t_i(X) - f_i^*\|^2). \tag{37}$$

$t_i(x)$ $x_1, x_2, \ldots, x_n$, . ., x,
:

$$t_i(x) = P_0 + P_1 x + P_2 x^2 + \cdots + P_k x^k,$$

$k \geqslant 0$, P_i $d_i \times n^i$, x^i
$x_{j_1} x_{j_2} \cdots x_{j_i}$, x_{j_p}
$x_1, x_2, \ldots, x_n$. I $\|x^i\| \leqslant \sqrt{n}\|x\|^i$ $i = 0, 1, \ldots, k$.
B ,

$$
\begin{aligned}
t_i(x) &= t_i(x - m_i^* + m_i^*) \\
&= P_0' + P_1'(x - m_i^*) + P_2'(x - m_i^*)^2 + \cdots + P_k'(x - m_i^*)^k,
\end{aligned} \tag{38}
$$

P_i' $d_i \times n^i$, $m_{i1}^*, \ldots, m_{in}^*$.

$$
\mathrm{f}_i^* = E_{\mathrm{f}_i^*}(t_i(X)) = P_0' + E_{\mathrm{f}_i^*}(P_1'(X - m_i^*)) + \cdots + E_{\mathrm{f}_i^*}(P_k'(X - m^*)^k) \tag{39}
$$

$$
E_{\mathrm{f}_i^*}(P_1'(X - m_i^*)) = P_1' E_{\mathrm{f}_i^*}(X - m_i^*) = 0,
$$

$$
t_i(X) - \mathrm{f}_i^* = \sum_{j=1}^{k} [P_j'(X - m_i^*)^j - E_{\mathrm{f}_i^*}(P_j'(X - m_i^*)^j)].
$$

N ,

$$
\begin{aligned}
E_{\mathrm{f}_i^*}(\|t_i(X) - \mathrm{f}_i^*\|^2) &= E_{\mathrm{f}_i^*}(\|(t_i(X) - \mathrm{f}_i^*)\ (t_i(X) - \mathrm{f}_i^*)\|) \\
&= E_{\mathrm{f}_i^*}\left(\left\| \sum_{j_1=1, j_2=1}^{k} [P_{j_1}'(X - m_i^*)^{j_1} - E_{\mathrm{f}_i^*}(P_{j_1}'(X - m_i^*)^{j_1})] \right.\right. \\
&\qquad \left.\left. \times [P_{j_2}'(X - m_i^*)^{j_2} - E_{\mathrm{f}_i^*}(P_{j_2}'(X - m_i^*)^{j_2})] \right\|\right) \\
&\leqslant \sum_{j_1=1, j_2=1}^{k} E_{\mathrm{f}_i^*}(\|[P_{j_1}'(X - m_i^*)^{j_1} - E(P_{j_1}'(X - m_i^*)^{j_1})]\ \| \\
&\qquad \times \|[P_{j_2}'(X - m_i^*)^{j_2} - E_{\mathrm{f}_i^*}(P_{j_2}'(X - m_i^*)^{j_2})]\|) \\
&\leqslant \sum_{j_1=1, j_2=1}^{k} E_{\mathrm{f}_i^*}^{1/2}(\|P_{j_1}'(X - m_i^*)^{j_1} - E_{\mathrm{f}_i^*}(P_{j_1}'(X - m_i^*)^{j_1})\|^2) \\
&\qquad \times E_{\mathrm{f}_i^*}^{1/2}\|P_{j_2}'(X - m_i^*)^{j_2} - E_{\mathrm{f}_i^*}(P_{j_2}'(X - m_i^*)^{j_2}|\,\mathrm{f}_i^*)\|^2).
\end{aligned} \tag{40}
$$

P ,

$$
\begin{aligned}
&E_{\mathrm{f}_i^*}(\|P_{j_1}'(X - m_i^*)^{j_1} - E_{\mathrm{f}_i^*}(P_{j_1}'(X - m_i^*)^{j_1})\|^2) \\
&\quad = E_{\mathrm{f}_i^*}(\|P_{j_1}'(X - m_i^*)^{j_1}\|^2) - \|E_{\mathrm{f}_i^*}(\|P_{j_1}'(X - m_i^*)^{j_1})\|^2 \\
&\quad \leqslant E_{\mathrm{f}_i^*}(\|P_{j_1}'(X - m_i^*)^{j_1}\|^2) \leqslant \sqrt{n} E_{\mathrm{f}_i^*}(\|P_{j_1}'\|^2 \|X - m_i^*\|^{2j_1}) \\
&\quad = \sqrt{n}\|P_{j_1}'\|^2 E_{\mathrm{f}_i^*}(\|X - m_i^*\|^{2j_1}).
\end{aligned} \tag{41}
$$

B $P_i(x|\,\mathrm{f}_i^*) \leqslant U_i(x|\,\mathrm{f}_i^*)$,

$$E_{\mathrm{f}_i^*}(\|X - m_i^*\|^{2j_1}) \leqslant \int \|x - m_i^*\|^{2j_1} U_i(x|\,\mathrm{f}_i^*) \quad x$$
$$= \int \|y\|^{2j_1} w(y + m_i^*)(1^i \quad)^{-c_1} \quad {-\mathrm{r}(1/(1^i \quad)^{\mathrm{n}c_2})\|y\|^{c_2}} \quad y, \tag{42}$$

$y = x - m_i^*$. $w(x)$,

$$w(y + m_i^*) \leqslant w_0 + w_1\|y\| + \cdots + w_{k'}\|y\|^{k'}, \tag{43}$$

k' , $w_0, w_1, \ldots, w_{k'}$
$\|m_i^*\|$, . .,

$$w_i = w_0^i + w_1^i\|m_i^*\| + \cdots + w_{c_i}^i\|m_i^*\|^{c_i} \qquad i = 0, 1, \ldots, k', \tag{44}$$

$w_0^i, w_1^i, \ldots, w_{c_i}^i$, $c_0, \ldots, c_{k'}$
. B L 1,

$$w_i \leqslant v_0^i + v_1^i\|m_i^* - m_j^*\| + \cdots + v_{c_i}^i\|m_i^* - m_j^*\|^{c_i} \qquad i = 0, 1, \ldots, k', \tag{45}$$

$v_0^i, v_1^i, \ldots, v_{c_i}^i$. $w(y + m_i^*)$
E . (42),

$$E_{\mathrm{f}_i^*}(\|X - m_i^*\|^{2j_1}) \leqslant \sum_{l=0}^{k'} w_l(1^i \quad)^{-c_1} \int \|y\|^{2j_1+l} \quad {-\mathrm{r}(1/(1^i \quad)^{\mathrm{n}c_2})\|y\|^{c_2}} \quad y$$
$$= \sum_{l=0}^{k'} w_l(1^i \quad)^{-c_1+\mathrm{n}(2j_1+l+1)} \int \|u\|^{2j_1+l} \quad {-\mathrm{r}\|u\|^{c_2}} \quad u,$$

$u = y/(1^i \quad)^{\mathrm{n}}$. C , $\int \|u\|^{2j_1+l} \quad {-\mathrm{r}\|u\|^{c_2}} \quad u$
j_1 . 1^i ,
$E_{\mathrm{f}_i^*}(\|X - m_i^*\|^{2j_1})$ $\|m_i^* - m_j^*\|$.
M , P'_{j_1} $m_{i1}^*, \ldots, m_{in}^*$, $\|P'_{j_1}\|$
$\|m_i^*\|$. , $E_{\mathrm{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1}\|^2)$
$\|m_i^* - m_j^*\|$.
A , $E_{\mathrm{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1} - E_{\mathrm{f}_i^*}(P'_{j_1}(X - m_i^*)^{j_1})\|^2)$
$\|m_i^* - m_j^*\|$. $\|m_i^* - m_j^*\| \geqslant T'$,

$$E_{\mathrm{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1} - E_{\mathrm{f}_i^*}(P'_{j_1}(X - m_i^*)^{j_1})\|^2) \leqslant \mathrm{c}_{j_1}\|m_i^* - m_j^*\|^{p_{j_1}}, \tag{46}$$

c_{j_1} p_{j_1} .
E . (46) E . (40),

$$E_{\mathrm{f}_i^*}(\|t_i(X) - \mathrm{f}_i^*\|^2) \leqslant \mathrm{c}\,\|m_i^* - m_j^*\|^{p}, \tag{47}$$

c p . , E . (37), ( )
.

A ( ), $j \neq i$, $f'_j = E_{f^*_j}(t_i(X))$

$$\begin{aligned} E_{f^*_j}(\|t_i(X) - f^*_i\|^2) &\leqslant E_{f^*_j}((\|t_i(X) - f'_j\| + \|f'_j - f^*_i\|)^2) \\ &= E_{f^*_j}(\|t_i(X) - f'_j\|^2 + 2\|t_i(X) - f'_j\|\|f'_j - f^*_i\| + \|f'_j - f^*_i\|^2) \\ &\leqslant E_{f^*_j}(2\|t_i(X) - f'_j\|^2 + 2\|f'_j - f^*_i\|^2) \\ &= 2E_{f^*_j}(\|t_i(X) - f'_j\|^2) + 2\|f^*_i - f'_j\|^2. \end{aligned} \tag{48}$$

I ,

$$E_{f^*_j}(\|t_i(X) - f'_j\|^2) \leqslant c_1\|m^*_i - m^*_j\|^{p_1}, \tag{49}$$

c_1 p_1 . M ,

$$\|f^*_i - f'_j\| \leqslant \|f^*_i\| + \|f'_j\|.$$

B E . (38), $\|f^*_i\|$ $\|f'_j\|$ $c_2\|m^*_i - m^*_j\|^{p_2}$, c_2 p_2 . , E . (48), $E_{f^*_j}(\|t_i(X) - f^*_i\|^2)$ $\|m^*_i - m^*_j\|$. $\|m^*_i - m^*_j\| \geqslant T'$,

$$E_{f^*_j}(\|t_i(X) - f^*_i\|^2) \leqslant c_j\|m^*_i - m^*_j\|^{p_j}, \quad j \neq i, \tag{50}$$

c_j p_j .

B E . (47) (50),

$$E(\|t_i(X) - f^*_i\|^2) = \sum_{j=1}^{K} a^*_j E_{f^*_j}(\|t_i(X) - f^*_i\|^2) \leqslant u M^q_i(F^*),$$

$M_i(F^*) =$ $_{j \neq i}\|m^*_i - m^*_j\|$, u q . □

Proof of Lemma 3.

$f(Z) = o(Z^p)$,

$Z \to 0$, p .

K

F^* $Z(F^*) = Z$. $i \neq j$, Z,

m^*_{ij} m^*_i m^*_j

$$a^*_i P_i(m^*_{ij}|f^*_i) = a^*_j P_j(m^*_{ij}|f^*_j).$$

$$E_i = \{x : a^*_i P_i(x|f^*_i) \geqslant a^*_j P_j(x|f^*_j)\},$$

$$E_j = \{x : a^*_j P_j(x|f^*_j) > a^*_i P_i(x|f^*_i)\}.$$

A $Z(F^*)$ $\boldsymbol{z}$, $(1^i\ )^n/(\|m^*_i - m^*_j\|)$ $(1^j\ )^n/(\|m^*_i - m^*_j\|)$. M , $k(S^*_i)$ $k(S^*_j)$. , (. .,) m^*_i (m^*_j) E_i (E_j). F , $\mathcal{N}_{r_i}(m^*_i)$ $\mathcal{N}_{r_j}(m^*_j)$ E_i E_j, , r_i r_j . $k(S^*_i)$ $k(S^*_j)$

, r_i r_j $\|m_i^* - m_j^*\|$ $\|m_i^* - m_j^*\|$. b_1

b_2

$$r_i \geqslant b_i \|m_i^* - m_j^*\| \qquad r_j \geqslant b_j \|m_i^* - m_j^*\|.$$

$$\mathscr{D}_i = \mathscr{N}^c_{r_i}(m_i^*) = \{x\colon \|x - m_i^*\| \geqslant r_i\},$$
$$\mathscr{D}_j = \mathscr{N}^c_{r_j}(m_j^*) = \{x\colon \|x - m_j^*\| \geqslant r_j\}$$

$$E_i \subset D_j, \qquad E_j \subset D_i.$$

M , $e_{ij}(\mathrm{F}^*)$ $h_k(x)$

$$\begin{aligned}
e_{ij}(\mathrm{F}^*) &= \int h_i(x)h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} \\
&= \int_{E_i} h_i(x)h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} + \int_{E_j} h_i(x)h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} \\
&\leqslant \int_{\mathscr{D}_j} h_i(x)h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} + \int_{\mathscr{D}_i} h_i(x)h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} \\
&\leqslant \int_{\mathscr{D}_j} h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} + \int_{\mathscr{D}_i} h_i(x)P(x|\mathrm{F}^*)\ \mathrm{m} \\
&= \mathrm{a}_j^* \int_{\mathscr{D}_j} P_j(x|\,\mathrm{f}_j^*)\ \mathrm{m} + \mathrm{a}_i^* \int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*)\ \mathrm{m}.
\end{aligned}$$

$\int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*)\ \mathrm{m}.$ $r_i \geqslant b_i \|m_i^* - m_j^*\|,$

$$\int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*)\ \mathrm{m} \leqslant \int_{\|x-m_i^*\| \leqslant b_i \|m_i^*-m_j^*\|} P_i(x|\,\mathrm{f}_i^*)\ \mathrm{m}.$$

B $y = (x - m_i^*)/\|m_i^* - m_j^*\|,$

$$\begin{aligned}
&\int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*)\ \mathrm{m} \\
&\quad \leqslant \int_{\|y\| \leqslant b_i} w(\|m_i^* - m_j^*\|y + m_i^*)(1^i\quad)^{-c_1}\ ^{-\mathrm{r}(\|m_i^*-m_j^*\|^{c_2})/(1\quad)^{\mathrm{n}c_2}\|y\|^{c_2}} \|m_i^* - m_j^*\|\ \mathrm{m}' \\
&\quad = \int_{\|y\| \leqslant b_i} \|m_i^* - m_j^*\| w(\|m_i^* - m_j^*\|y + m_i^*)(1^i\quad)^{-c_1} \\
&\qquad \times\ ^{-\mathrm{r}(\|m_i^*-m_j^*\|^{c_2})/(1\quad)^{\mathrm{n}c_2}\|y\|^{c_2}}\ \mathrm{m}',
\end{aligned} \tag{51}$$

m' m .

$w(\|m_i^* - m_j^*\|y + m_i^*)$

m_i^* $\|m_i^* - m_j^*\|,$

q

$\|m_i^* - m_j^*\|^{-q} w(\|m_i^* - m_j^*\|y + m_i^*)$ $\mathrm{Z}(\mathrm{F}^*) \to 0.$

$\|m_i^* - m_j^*\|^{-q} w(\|m_i^* - m_j^*\| y + m_i^*)$
y . M , L 1,

$$\|m_i^* - m_j^*\|^{1+q}(1^i \quad)^{-c_1} \leqslant O(\mathrm{Z}^{-c_1'}),$$
$$\|m_i^* - m_j^*\|^{c_2}(1^i \quad)^{-\mathrm{n}c_2} \geqslant O(\mathrm{Z}^{-c_2}),$$

$c_1' = (q+1) \vee (c_1/\mathrm{n})$.
A , E . (51)

$$\int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*) \quad \mathrm{m} \leqslant \int_{\mathscr{B}_i} \frac{1}{\mathrm{Z}^{c_1'}(\mathrm{F}^*)} w_1(y) \quad ^{-\mathrm{r}'(1/\mathrm{Z}^{c_2}(\mathrm{F}^*))\|y\|^{c_2}} \quad \mathrm{m}'$$
$$= \int_{\mathscr{B}_i} \frac{1}{\mathrm{Z}^{c_1'}} w_1(y) \quad ^{-\mathrm{r}'(1/\mathrm{Z}^{c_2})\|y\|^{c_2}} \quad \mathrm{m}', \tag{52}$$

$\mathscr{B}_i = \{y\colon \|y\| \geqslant b_i\}$, r' , $w_1(y)$
y .
F ,

$$F_i(\mathrm{Z}) = \int_{\mathscr{B}_i} P(y|\mathrm{Z}) \quad y, \quad P(y|\mathrm{Z}) = \frac{1}{\mathrm{Z}^{c_1'}} w_1(y) \quad ^{-\mathrm{r}'(1/\mathrm{Z}^{c_2})\|y\|^{c_2}}$$

$F_i(\mathrm{Z})/\mathrm{Z}^p$ Z $\boldsymbol{z}$.
F $y \in \mathscr{B}_i$,

$$\underset{\mathrm{Z}\to 0}{} \frac{P(y|\mathrm{Z})}{\mathrm{Z}^p} = w_1(y) \underset{\mathrm{Z}\to 0}{} \frac{1}{\mathrm{Z}^{(c_1'+p)}} \quad ^{-\mathrm{r}'(1/\mathrm{Z}^{c_2})\|y\|^{c_2}}$$
$$= w_1(y) \underset{\mathrm{z}=\frac{1}{\mathrm{Z}}\to\infty}{} \frac{\mathrm{z}^{(c_1'+p)}}{\quad ^{\mathrm{z}^{c_2}\,\mathrm{r}'\|y\|^{c_2}}}$$
$$= 0,$$

$\mathscr{B}_i$,

$$\underset{\mathrm{Z}\to 0}{} \frac{F_i(\mathrm{Z})}{\mathrm{Z}^p} = \underset{\mathrm{Z}\to 0}{} \int_{\mathscr{B}_i} \frac{P(y|\mathrm{Z})}{\mathrm{Z}^p} \quad \mathrm{m}'$$
$$= \int_{\mathscr{B}_i} \underset{\mathrm{Z}\to 0}{} \frac{P(y|\mathrm{Z})}{\mathrm{Z}^p} \quad \mathrm{m}'$$
$$= 0$$

$F_i(\mathrm{Z}) = o(\mathrm{Z}^p)$. I E . (52)

$$\underset{\mathrm{Z}(\mathrm{F}^*)=\mathrm{Z}}{} \int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*) \quad \mathrm{m} = o(\mathrm{Z}^p). \tag{53}$$

, :

$$\underset{\mathrm{Z}(\mathrm{F}^*)=\mathrm{Z}}{} \int_{\mathscr{D}_j} P_j(x|\,\mathrm{f}_j^*) \quad \mathrm{m} = o(\mathrm{Z}^p).$$

A ,

$$f_{ij}(Z) = {}_{Z(F^*)=Z} e_{ij}(F^*)$$
$$\leqslant {}_{Z(F^*)=Z} \left(a_j^* \int_{\mathscr{D}_j} P_j(x|\, f_j^*) \quad m + a_i^* \int_{\mathscr{D}_i} P_i(x|\, f_i^*) \quad m \right)$$
$$\leqslant {}_{Z(F^*)=Z} \int_{\mathscr{D}_j} P_j(x|\, f_j^*) \quad x + {}_{Z(F^*)=Z} \int_{\mathscr{D}_i} P_i(x|\, f_i^*) \quad m)$$
$$= o(Z^p).$$

,

$$f(Z) \leqslant {}_{ij} \quad f_{ij}(Z) = o(Z^p). \tag{54}$$

M ,

$$ {}_{Z\to 0} \frac{f^e(Z)}{Z^p} = {}_{Z\to 0} \left(\frac{f(Z)}{Z^{\frac{p}{e}}} \right)^e = 0,$$

$f^e(Z) = o(Z^p)$ $f^e(Z(F^*)) = o(Z^p(F^*))$. □

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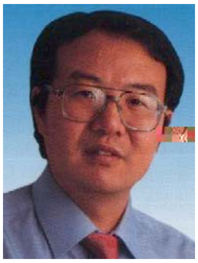

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