1. Introduction

EM (ML) (MAP) [3]. EM (. ., [2,4,12,15,16,18,19]). G , EM . , EM A [9], [5], -N [6,10], [13] ” [7]. H , EM - M [17], [7] [8], - (. .,). I , EM .

J [20] H EM G H , EM . M , , EM , . , , EM -N ML MAP .

I M . [14] EM G z . B EM J [20], z . , EM z .

I , EM [18], [14] G . , EM z . F , [14] G .

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EM . z

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2. The EM algorithm for mixtures of densities from exponential families

2.1. The mixture model

:

$$P(x|\Phi) = \sum_{i=1}^{K} \alpha_i P_i(x|\phi_i), \quad \alpha_i \geqslant 0, \ \sum_{i=1}^{K} \alpha_i = 1, \tag{1}$$

$x = [x_1, \ldots, x_n] \in R^n$, P_i

z $\phi_i \in \Omega_i \subset R^{d_i}$, K

. Φ α_i

ϕ_i, , $\Phi = (\alpha_1, \ldots, \alpha_K, \phi_1, \ldots, \phi_K) \in \Omega$,

$$\Omega = \left\{ (\alpha_1, \ldots, \alpha_K, \phi_1, \ldots, \phi_K) : \sum_{i=1}^{K} \alpha_i = 1 \quad \alpha_i \geqslant 0, \ \phi_i \in \Omega_i \quad i = 1, \ldots, K \right\}.$$

I $P_i(x|\phi_i) = P_i(x|m_i, \Sigma_i)$ G

$$P_i(x|\phi_i) = P(x|m_i, \Sigma_i) = \frac{1}{(2\pi)^{n/2}(\quad \Sigma_i)^{1/2}} \quad {}^{-(1/2)(x-m_i) \ \Sigma_i^{-1}(x-m_i)}, \tag{2}$$

$m_i = [m_{i1}, \ldots, m_{in}]$, $\Sigma_i = (\sigma^i_{kl})_{n \times n}$

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G .

A $q(x|\theta), \theta \in \Theta \subset R^d$ R^n

$$q(x|\theta) = a(\theta)^{-1} b(x) \quad {}^{\theta \ t(x)}, \quad x \in R^n, \tag{3}$$

$b(x), t(x)$ x R^n $a(\theta)$

$$a(\theta) = \int_{R^n} b(x) \quad {}^{\theta \ t(x)} \ \mu$$

μ R^n. I $b(x) \geqslant 0$

$x \in R^n$, $a(\theta) < +\infty$ $\theta \in \Theta$ $t(x)$, ,

. A ,

$b(x)$.

O ● , ● ●

$$P(x|\text{f}) = q(x|\text{y}(\text{f})) = a(\text{f})^{-1}b(x) \quad {}^{\text{y}(\text{f}) \ \ t(x)}, \quad x \in R^n$$

● ” $\text{f} = E_{\text{y}}(t(X))$ ●
(● 1] ● 18] ●).
● ● ● ●
$P(x|\text{f})$ ● m ● S,
● ● ● ● $U(x|\text{f})$ ● ● :

$$P(x|\text{f}) \leqslant U(x|\text{f}) = w(x)(1 \quad)^{-c_1} \ -\text{r}(1/\text{Z}(x))^{-c_2}, \tag{4}$$

$$\text{Z}(x) = \frac{(1 \quad)^{\text{n}}}{\|x - m\|}$$

l ● S ● $P(x|\text{f})$. M● ● , c_1, c_2, r n
● ● ● , $w(x)$ ● ● ● ● ● $x_1, \ldots, x_n$
● ● . H , E ● ●
● ● ● . A , ●
● ● ● ● ● ● ,
G ● ● .
I , $P_i(x|\text{f}_i)$ ● ● ” z -
● ● $\text{f}_i \in \text{O}_i \subset R^{d_i}$ ● ● :

$$P_i(x|\text{f}_i) = a_i(\text{f}_i)^{-1}b_i(x) \quad {}^{\text{y}_i(\text{f}_i) \ \ t_i(x)}, \quad x \in R^n \tag{5}$$

$\text{F}^* = (\text{a}_1^*, \ldots, \text{a}_K^*, \text{f}_1^*, \ldots, \text{f}_K^*)$ ● ● ” ●
● , , ● ● ●
F^*. A ●, ● ● ● $t_i(x)$ ● ● ● ●
$x_1, \ldots, x_n$.
M● ● , ● ● ● $P_i(x|\text{f}_i^*)$ ● ● :

$$P_i(x|\text{f}_i^*) \leqslant U_i(x|\text{f}_i^*) = w(x)(1^i \quad)^{-c_1} \ -\text{r}(1/\text{Z}_i(x))^{c_2}, \tag{6}$$

$$\text{Z}_i(x) = \frac{(1^i \quad)^{\text{n}_i}}{\|x - m_i^*\|}$$

m_i^* 1^i ● ● ●
S_i^* ● $P_i(x|\text{f}_i^*)$, . c_1, c_2, r $w(x)$ ● i .
F ● , 1^i ● . n ●
● ● n_i ● r $U_i(x|\text{f}_i^*)$ ●
● ● ● ● . A ,

$$\text{Z}_i(x) = \frac{(1^i \quad)^{\text{n}}}{\|x - m_i^*\|}, \quad i = 1, \ldots, K,$$

n ● ● ● .

2.2. The EM algorithm and its asymptotic convergence rate

• • $\mathscr{S}_N = \{x^{(t)} : t = 1, \ldots, N\}$ •
f $_1, \ldots,$ f $_K$. I F $= (\mathrm{a}_1, \ldots, \mathrm{a}_K,$ f $_1, \ldots,$ f $_K)$
ML • • - •• • $L(\mathrm{F}) = \sum_{t=1}^{N}$ • $P(x^{(t)}|\mathrm{F})$ •
• • • E . (1), EM •

$$\mathrm{a}_i^+ = \frac{1}{N}\sum_{t=1}^{N} \frac{\mathrm{a}_i P_i(x^{(t)}|\,\mathrm{f}_i)}{P(x^{(t)}|\mathrm{F}\ )}, \tag{7}$$

$$\mathrm{f}_i^+ = \left\{\sum_{t=1}^{N} t_i(x^{(t)}) \frac{\mathrm{a}_i P_i(x^{(t)}|\,\mathrm{f}_i)}{P(x^{(t)}|\mathrm{F}\ )}\right\} \Bigg/ \left\{\sum_{t=1}^{N} \frac{\mathrm{a}_i P_i(x^{(t)}|\,\mathrm{f}_i)}{P(x^{(t)}|\mathrm{F}\ )}\right\}, \tag{8}$$

• $i = 1, \ldots, K$.
• • • • • L(F) 3,19]. M• • ,
• • , EM • • • •
• • z • •• L(F) 18]. I ,
EM • • • • F* • (. .,
z N • $\mathscr{S}_N$, EM • • • F^N
$_{N\to\infty}$F^N = F* • •), z • •
• • F*.
I 18], EM • • •
F$^+ = G$(F)

$$\mathrm{F}^+ - \mathrm{F}^N = G(\mathrm{F}\ ) - G(\mathrm{F}^N) = G'(\mathrm{F}^N)(\mathrm{F}\ - \mathrm{F}^N) + O(\|\mathrm{F}\ - \mathrm{F}^N\|^2) \tag{9}$$

• F O F^N, G'(F) • J • • G(F) F^N $O(x)$
• $x \to 0$. B • ,
• N • , • • , G'(F^N) • •
• $E(G'(\mathrm{F}^*)) = I - Q(\mathrm{F}^*)R(\mathrm{F}^*)$,

$$Q(\mathrm{F}^*) = diag(\mathrm{a}_1^*, \ldots, \mathrm{a}_K^*, \mathrm{a}_1^{*-1}\,\mathrm{P}_1, \ldots, \mathrm{a}_K^{*-1}\,\mathrm{P}_K) \tag{10}$$

$$\mathrm{P}_i = \int_{R^n} [t_i(x) - \mathrm{f}_i^*][t_i(x) - \mathrm{f}_i^*]\ P_i(x|\,\mathrm{f}_i^*)\ \mathrm{m}$$

$$R(\mathrm{F}^*) = \int_{R^n} V(x)V(x)\ P(x|\mathrm{F}^*)\ \mathrm{m} \tag{11}$$

$$V(x) = (\mathrm{b}_1(x), \ldots, \mathrm{b}_K(x), \mathrm{a}_1^*\mathrm{b}_1(x)\mathrm{G}_1(x)\ , \ldots, \mathrm{a}_K^*\mathrm{b}_K(x)\mathrm{G}_K(x)\ )\ ,$$

$$\mathrm{b}_i(x) = P_i(x|\,\mathrm{f}_i^*)/P(x|\mathrm{F}^*),$$

$$\mathrm{G}_i(x) = \mathrm{P}_i^{-1}[t_i(x) - \mathrm{f}_i^*].$$

H , $E(\cdot) = E_{\Phi^*}(\cdot)$. I ● ● ● E . (9) ●
● EM ● ● ● Φ^N ● ● $\|G'(\Phi^N)\|$. B
N ● , ● ● ● ● ●
● r ● EM ● ● ● Φ^*:

$$r \leqslant \lim_{N\to\infty} \|G'(\Phi^N)\| = \left\| \lim_{N\to\infty} G'(\Phi^N) \right\|$$

$$= \|E(G'(\Phi^*))\| = \|I - Q(\Phi^*)R(\Phi^*)\|. \tag{12}$$

I ● ● , ● ● ● EM
● ● ● ● ●
● ● ● ● ● ●
z ●
● z ●.

3. The main result

3.1. The measures of the overlap

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G . ● ● ● ● ● ● E .
(1) Φ^*:

$$h_i(x) = \frac{\alpha_i^* P_i(x|\phi_i^*)}{\sum_{j=1}^K \alpha_j^* P_j(x|\phi_j^*)} \quad ● \quad i = 1, \ldots, K. \tag{13}$$

I ● ● ● E . (11)

$$h_i(x) = \alpha_i^* \beta_i(x). \tag{14}$$

$$\gamma_{ij}(x) = (\delta_{ij} - h_i(x))h_j(x) \quad ● \quad i,j = 1, \ldots, K, \tag{15}$$

δ_{ij} K ● ● . , ● ● ●
● ● ● ● ● ● :

$$e_{ij}(\Phi^*) = \int_{R^n} |\gamma_{ij}(x)| P(x|\Phi^*) \, m$$

● $i,j = 1, 2, \ldots, K$, $e_{ij}(\Phi^*) \leqslant 1$ $|\gamma_{ij}(x)| \leqslant 1$.
F● $i \neq j$, $e_{ij}(\Phi^*)$ ● ● ●
● ● ● i j . $P_i(x|\phi_i^*)$ $P_j(x|\phi_j^*)$
● ● x, $h_i(x)h_j(x)$

$e(\Phi^*) = 0$

$h_i(x)h_j(x) = 0$ $i \neq j$

, . ., .

I , 18] z EM N z . H , . I , , $e(\Phi^*)$ z . A EM z , $e(\Phi^*)$.

3.2. Regular conditions and lemmas

:

(1) *Nondegenerate condition on the mixing proportions*:

:

$$\alpha_i^* \geqslant \alpha \quad i = 1, \ldots, K, \tag{16}$$

α . I z , , . .

(2) *Uniform attenuating condition on the eigenvalues of the covariance matrices*:

Σ_i^* i $\lambda_{i1}, \ldots, \lambda_{in}$.

$$\beta\lambda(\Phi^*) \leqslant \lambda_{ij} \leqslant \lambda(\Phi^*) \quad i = 1, \ldots, K, \ k = 1, \ldots, n, \tag{17}$$

β $\lambda(\Phi^*)$ $\Sigma_1^*, \ldots, \Sigma_K^*$, . .,

$$\lambda(\Phi^*) = \quad \lambda_{ij}$$
i,j

B. , z z . I E . (17) K , . .,

$$1 \leqslant \kappa(\Sigma_i^*) \leqslant B' \quad i = 1, \ldots, K,$$

$\kappa(\Sigma_i^*)$ Σ_i^* B' .

(3) *Regular condition on the mean vectors*: ●

● ● ● ● , . ., $m_1^*, \ldots, m_K^*$,

mD $(F^*) \leqslant D$ $(F^*) \leqslant \|m_i^* - m_j^*\| \leqslant D$ (F^*) ● $i \neq j$, (18)

D $(F^*) =$ ${}_{i \neq j}\|m_i^* - m_j^*\|$, D $(F^*) =$ ${}_{i \neq j}\|m_i^* - m_j^*\|$, m

● . , ● ●

● ● . M● ● , ● ●

●z ●, ● ● m_i^*, m_j^* ● ● , . .,

● ● T $\|m_i^* - m_j^*\| \geqslant T$ $i \neq j$. I ● ,

E . (18) ● $m_1^*, \ldots, m_K^*$ ● .

● ● $\| P_i^{-1} \|$, P_i E . (10).

● ● $P_i(x| f_i^*)$ ● ● ● , ● ●

11]:

$P_i \Rightarrow x$

Z(F*) → 0. • , Z(F*) → 0 e(F*) → 0 Z(F*) → 0 • Z(F*) → 0 • .
A • • • • • •
• , • e(F*) •
z •, • Z(F*) → 0 e(F*) → 0 • .
Z(F*) • • , . ., F* • •
Z(F*),

$$f(Z) = \underset{Z(F^*)=Z}{} e(F^*) \tag{21}$$

e(F*) • 1. B • ,

$$e_{ij}(F^*) \leqslant e(F^*) \leqslant f(Z(F^*)) \quad \bullet \quad i \neq j. \tag{22}$$

F , • • (A A • •).

Lemma 1. *Suppose that a mixture of K densities from the bell sheltered exponential families of the parameter* F* *satisfies Conditions* (1) (3). *As* Z(F*) *tends to zero*, *we have*

( ) Z(F*), $Z_i(m_j^*)$ *and* $Z_j(m_i^*)$ *are the equivalent infinitesimals.*
( ) *For* $i \neq j$, *we have*

$$\|m_i^*\| \leqslant T' \|m_i^* - m_j^*\|, \tag{23}$$

where T' *is a positive number.*
( ) *For any two nonnegative numbers with* $p + q > 0$, *we have*

$$\|m_i^* - m_j^*\|^p (1^i \quad)^{-nq} \leqslant O(Z^{-p \vee q}(F^*)), \tag{24}$$

where $p \vee q =$ $\{p, q\}$.

Lemma 2. *Suppose that a mixture of K densities from the bell sheltered exponential families of the parameter* F* *satisfies Conditions* (1) (3). *As* Z(F*) *tends to zero*, *we have for each i*

( ) $$\| P_i \| \leqslant c \|m_i^* - m_j^*\|^p, \tag{25}$$

where $j \neq i$, c *and p are some positive numbers.*

( ) $$E(\|t_i(X) - f_i^*\|^2) \leqslant u M_i^q(F^*), \tag{26}$$

where $M_i(F^*) = \quad_{j \neq i} \|m_i^* - m_j^*\|$, u *and q are some positive numbers.*

Lemma 3. *Suppose that a mixture of K densities from the bell sheltered exponential families of the parameter* F* *satisfies Conditions* (1) (3) *and* Z(F*) → 0 *as an infinitesimal*, *we have*

$$f^e(Z(F^*)) = o(Z^p(F^*)), \tag{27}$$

where $\varepsilon > 0$, *p is any positive number and* $o(x)$ *means that it is a higher order infinitesimal as* $x \to 0$.

• • •• • • . E , • $e(\Phi^*)$ $Z(\Phi^*)$ L 3 • • • .

3.3. The main theorem

• • , • • • .

Theorem 1. *Given a mixture of K densities from the bell sheltered exponential families of the parameter* Φ^* *that satisfies Conditions* (1) (4), *as* $e(\Phi^*)$ *tends to zero as an infinitesimal, we have*

$$r \leqslant \|E(G'(\Phi^*))\| = o(\ ^{0.5-\varepsilon}(\Phi^*)), \tag{28}$$

where ε *is an arbitrarily small positive number.*

A • • • , • • , • • • • • • • • , • • , $e(\Phi^*) \to 0$, $\|E(G'(\Phi^*))\|$ • $^{0.5-\varepsilon}(\Phi^*)$. • , (Φ^*) • z •, • • • EM • • • F* • $^{0.5-\varepsilon}(\Phi^*)$. , $e(\Phi^*)$ N • , • • EM • • • z •. I • • , EM • -N • • • . M• • , • • • • • • • • • z •. • , • • EM . • • • • • • EM • • . O • , • • • •• • - • • EM • • • - • • . A , • • • • • • • • - • • . • •z •, • • • • . , • • - • • •z •. • , • • • • - • • . I , • • • EM • • • • - • • • z •. M• • , • • • • •• • • • • • • ” • [16] P -EM • [13], • • EM • • • • • - • • .

Proof of Theorem 1. • • • • $Q(\mathrm{F}^*)R(\mathrm{F}^*)$.
A • • • • $Q(\mathrm{F}^*)$ $R(\mathrm{F}^*)$, • •
$Q(\mathrm{F}^*)R(\mathrm{F}^*)$ • , • • :

$$Q(\mathrm{F}^*)R(\mathrm{F}^*) = diag[diag[\mathscr{A}\ ], \mathrm{a}^{*-1}\,\mathrm{P}_1, \ldots, \mathrm{a}_K^{*-1}\,\mathrm{P}_K] \times \begin{pmatrix} R_{\mathrm{b,b}} & R_{\mathrm{b,G_1}} & \cdots & R_{\mathrm{b,G}_K} \\ R_{\mathrm{G_1,b}} & R_{\mathrm{G_1,G_1}} & \cdots & R_{\mathrm{G_1,G}_K} \\ \vdots & \vdots & \ddots & \vdots \\ R_{\mathrm{G}_K,\mathrm{b}} & R_{\mathrm{G}_K,\mathrm{G_1}} & \cdots & R_{\mathrm{G}_K,\mathrm{G}_K} \end{pmatrix}$$

$$= \begin{pmatrix} diag[\mathscr{A}\ ]R_{\mathrm{b,b}} & diag[\mathscr{A}\ ]R_{\mathrm{b,G_1}} & \cdots & diag[\mathscr{A}\ ]R_{\mathrm{b,G}_K} \\ \mathrm{a}_1^{*-1}\,\mathrm{P}_1 R_{\mathrm{G_1,b}} & \mathrm{a}_1^{*-1}\,\mathrm{P}_1 R_{\mathrm{G_1,G_1}} & \cdots & \mathrm{a}_1^{*-1}\,\mathrm{P}_1 R_{\mathrm{G_1,G}_K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{a}_K^{*-1}\,\mathrm{P}_K R_{\mathrm{G}_K,\mathrm{b}} & \mathrm{a}_K^{*-1}\,\mathrm{P}_K R_{\mathrm{G}_K,\mathrm{G_1}} & \cdots & \mathrm{a}_K^{*-1}\,\mathrm{P}_K R_{\mathrm{G}_K,\mathrm{G}_K} \end{pmatrix},$$

$\mathrm{b}(x) = [\mathrm{b}_1(x), \ldots, \mathrm{b}_K(x)]$ $\mathscr{A} = [\mathrm{a}_1^*, \ldots, \mathrm{a}_K^*]$. • •
$R(\mathrm{F}^*)$ • • • • $V(x)$

$$V(x) = [\mathrm{b}(x)\ , \mathrm{a}_1^*\mathrm{b}_1(x)\mathrm{G}_1(x)\ , \ldots, \mathrm{a}_K^*\mathrm{b}_K(x)\mathrm{G}_K(x)\ ]\ .$$

( ) *The computation of* $diag[\mathscr{A}\ ]R_{\mathrm{b,b}}$: F • • • $\mathrm{b}_i(x)$ •
$h_i(x) = \mathrm{a}_i^*\mathrm{b}_i(x)$,

$$\int_{R^n} \mathrm{b}_i(x)\mathrm{b}_j(x)P(x|\mathrm{F}^*)\ \ \mathrm{m} = \frac{1}{\mathrm{a}_i^*\mathrm{a}_j^*} e_{ij}(\mathrm{F}^*) \qquad i \neq j,$$

$$\int_{R^n} \mathrm{b}_i^2(x)P(x|\mathrm{F}^*)\ \ \mathrm{m} = \frac{1}{\mathrm{a}_i^*} - \frac{1}{(\mathrm{a}_i^*)^2} e_{ii}(\mathrm{F}^*)$$

•

$$diag[\mathscr{A}\ ]R_{\mathrm{b,b}} \ = I_K + \begin{pmatrix} -\mathrm{a}_1^{*-1}e_{11}(\mathrm{F}^*) & \mathrm{a}_2^{*-1}e_{12}(\mathrm{F}^*) & \cdots & \mathrm{a}_K^{*-1}e_{1K}(\mathrm{F}^*) \\ \mathrm{a}_1^{*-1}e_{21}(\mathrm{F}^*) & -\mathrm{a}_2^{*-1}e_{22}(\mathrm{F}^*) & \cdots & \mathrm{a}_K^{*-1}e_{2K}(\mathrm{F}^*) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{a}_1^{*-1}e_{K1}(\mathrm{F}^*) & \mathrm{a}_2^{*-1}e_{K2}(\mathrm{F}^*) & \cdots & -\mathrm{a}_K^{*-1}e_{KK}(\mathrm{F}^*) \end{pmatrix}.$$

B

$$\frac{1}{\mathrm{a}_j^*} e_{ij}(\mathrm{F}^*) \leqslant \frac{1}{\mathrm{a}} e_{ij}(\mathrm{F}^*) = o(\ ^{0.5-\mathrm{e}}(\mathrm{F}^*)),$$

$diag[\mathscr{A}\ ]R_{\mathrm{b,b}}\ = I_K + o=$

, ● ● . I ● ● ●
C z $|g_{ij}(x)|\leqslant 1$

$$
\begin{aligned}
&|E(h_j(X)(h_i(X)-d_{ij})(t_{i,k}(X)-f^*_{i,k}))|\\
&\quad\leqslant E(|h_j(X)(h_i(X)-d_{ij})||(t_{i,k}(X)-f^*_{i,k})|)\\
&\quad\leqslant E^{1/2}(g^2_{ij}(X))E^{1/2}((t_{i,k}(X)-f^*_{i,k})^2)\\
&\quad\leqslant E^{1/2}(|g_{ij}(X))E^{1/2}((t_{i,k}(X)-f^*_{i,k})^2)\\
&\quad\leqslant \sqrt{e_{ij}(F^*)}E^{1/2}((t_{i,k}(X)-f^*_{i,k})^2).
\end{aligned}
$$

A ● ● L 2, $E(\|t_i(X)-f^*_i\|^2|F^*)$ ● $uM^q_i(F^*)$. , $E^{1/2}((t_{i,k}(X)-f^*_{i,k})^2)$ ● $\sqrt{uM^q_i(F^*)}$. ● ,

$$E(diag[\mathscr{A}\ ]a^*_i b_i(X)b(X)(t_i(X)-f^*_i)\ )=O(M^{q/2}_i(F^*)e^{0.5}(F^*)).$$

A ● ● L 1 3, $M^{q/2}_i(F^*)\ ^{0.5}(F^*)$ ● $e(F^*)$ ● $Z(F^*)$ ● z ●. I ● ● ● ● ● ●

$$\|E(diag[\mathscr{A}\ ]a^*_i b_i(X)b(X)(t_i(X)-f^*_i)\ )\|=O(M^{q/2}_i(F^*)\ ^{0.5}(F^*)).$$

M● ● ,

$$\|diag[\mathscr{A}\ ]R_{b,G_i}\|\leqslant\|E(diag[\mathscr{A}\ ]a^*_i b_i(X)b(X)(t_i(X)-f^*_i)\ )\|\|P^{-1}_i\|$$

$$\|P^{-1}_i\|=\|I(f^*_i)\|\leqslant O(\|m^*_i\|^{t_1}(1^i\quad)^{-t_2})$$

C● ● (4). ,

$$\|diag[\mathscr{A}\ ]R_{b,G_i}\|\leqslant u\|m^*_i-m^*_{j'}\|^{q_1}(1^i\quad)^{-q_2}\ ^{0.5}(F^*),$$

$q_1=(q/2)+t_1$, $q_2=t_2$, u ● .
● , ● ● ● L 1 3

$$\|diag[\mathscr{A}\ ]R_{b,G_i}\|\leqslant O(Z^{-q_1\vee q_2}(F^*))e^{0.5}(F^*)=o(e^{0.5-e}(F^*)).$$

B ● ● ● , ●

$$diag[\mathscr{A}\ ]R_{b,G_i}=o(\ ^{0.5-e}(F^*)).$$

( ) *The computation of* $a^{*-1}_i P_i R_{G_i,b}$ $(i=1,\ldots,K)$: A ● ● ● ● (1) L 2, $a^{*-1}_i\|P_i\|$ ● $(1/a)c\|m^*_i-m^*_j\|^p$, $j\neq i$, c p ● . B $R_{G_i,b}=R_{b,G_i}$, ( ) ● :

$$a^{*-1}_i P_i R_{G_i,b}\ =o(\ ^{0.5-e}(F^*)).$$

( ) *The computation of* $\alpha_i^{*-1}\Pi_i R_{\Gamma_i,\Gamma_i}$ $(i=1,\ldots,K)$: B ● ● $V(x)$,

$$\alpha_i^{*-1}\Pi_i R_{\Gamma_i,\Gamma_i} = \alpha_i^{*-1}\Pi_i E(h_i^2(X)\Gamma_i(X)\Gamma_i (X))$$
$$= \alpha_i^{*-1}E(h_i^2(X)(t_i(X)-\phi_i^*)(t_i(X)-\phi_i^*))\Pi_i^{-1}$$
$$= I_{d_i} + \alpha_i^{*-1}E(h_i(X)(h_i(X)-1)(t_i(X)-\phi_i^*)(t_i(X)-\phi_i^*))\Pi_i^{-1},$$

:

$$\Pi_i E(h_i(X)\Gamma_i(X)\Gamma_i (X)) = \alpha_i^* I_{d_i}.$$

F ● , ● $E(\|t_i(X)-\phi_i^*\|^2|\Phi^*)$ ● $\mathrm{u}M_i^q(\Phi^*)$ α_i^{*-1} ● , ● , ● :

$$\alpha_i^{*-1}E(h_i(X)(h_i(X)-1)(t_i(X)-\phi_i^*)(t_i(X)-\phi_i^*))\Pi_i^{-1} = o(\ ^{0.5-\varepsilon}(\Phi^*))$$

● ,

$$\alpha_i^{*-1}\Pi_i R_{\Gamma_i,\Gamma_i} = I_{d_i} + o(\ ^{0.5-\varepsilon}(\Phi^*)).$$

( ) *The computation of* $\alpha_i^{*-1}\Pi_i R_{\Gamma_i,\Gamma_j}$ $(j\neq i)$: B ● ● $V(x)$,

$$\alpha_i^{*-1}\Pi_i R_{\Gamma_i,\Gamma_j} = \alpha_i^{*-1}E(\alpha_i^*\beta_i(X)\alpha_j^*\beta_j(X)(t_i(X)-\phi_i^*)(t_j(X)-\phi_j^*))\Pi_j^{-1}$$
$$= \alpha_i^{*-1}E(h_i(X)h_j(X)(t_i(X)-\phi_i^*)(t_j(X)-\phi_j^*))\Pi_j^{-1}.$$

( ), ●

$$\alpha_i^{*-1}\Pi_i R_{\Gamma_i,\Gamma_j} = o(\ ^{0.5-\varepsilon}(\Phi^*)).$$

( ) ( ), ● :

$$Q(\Phi^*)R(\Phi^*) = I + o(\ ^{0.5-\varepsilon}).$$

, ● ● E . (12),

$$r \leqslant \|I - Q(\Phi^*)R(\Phi^*)\| = o(\ ^{0.5-\varepsilon}(\Phi^*)). \quad \square$$

4. A typical class: Gaussian mixtures

● ● ● EM ● ● G ● ● ● ● . A ● 1], G $P_i(x|m_i,\Sigma_i)$ E . (2) ● ● $\theta_i = (\Sigma_i^{-1}m_i, \Sigma_i^{-1})$ ● ● $t_i(x) = (x, -\frac{1}{2}xx\ )$. ● , ϕ_i, ● ● ● θ_i, $(m_i, -\frac{1}{2}(\Sigma_i + m_i m_i\ ))$ ● ● ● (m_i,Σ_i) ● ● .

N , • • G
- • • • •
• .

Lemma 4. *Suppose that* $P_i(x|\mathrm{f}_i^*) = P_i(x|m_i^*, \mathrm{S}_i^*)$ *is a Gaussian distribution with the mean* m_i^* *and the covariance matrix* S_i^*, *and that the condition number of* S_i^*, *i.e.*, $\mathrm{k}(\mathrm{S}_i^*)$, *is upper bounded by* B'. *We have that* $P_i(x|\hat{\mathrm{f}}_i^*)$ *is bell-sheltered, i.e.*,

$$P_i(x|\mathrm{f}_i^*) = P_i(x|m_i^*, \mathrm{S}_i^*) \leqslant b\frac{1}{(1^i \quad)^{n/2}} \quad {}^{-(1/21^i \quad)\|x-m_i^*\|^2}, \tag{29}$$

where b *is a positive number.*

Proof. B • • • • • $y = U_i(x - m_i^*)$ • •

$$P(y|1^i \quad) = \frac{1}{(2\mathrm{p}1^i \quad)^{n/2}} \quad {}^{-(1/21^i \quad)\|y\|^2},$$

$$P_i(x|m_i^*, \mathrm{S}_i^*) \leqslant B'^{n/2} P(y|1^i \quad),$$

$\mathrm{k}(\mathrm{S}_i^*) \leqslant B'$. M• • , • $\|y\| = \|x - m_i^*\|$,

$$P_i(x|m_i^*, \mathrm{S}_i^*) \leqslant b\frac{1}{(1^i \quad)^{n/2}} \quad {}^{-(1/21^i \quad)\|x-m_i^*\|^2},$$

$b = (B'/2\mathrm{p})^{n/2}$. □

B L 4, • • (1) (3), G • F*
• K • • .
M• • , • • • $P_i(x|\mathrm{f}_i^*) = P_i(x|m_i^*, \mathrm{S}_i^*)$, $t_i(x)$
• • • :

$$t_i(x) = \begin{cases} x & \bullet \; m_i^*, \\ -\frac{1}{2}xx & \bullet \; -\frac{1}{2}(\mathrm{S}_i^* + m_i^*(m_i^*) \;). \end{cases}$$

• • • $t_i(x)$ • • • $x_1, \ldots, x_n$.
• , G • F* • • (1) (3)
• • • • • • (4).
F• , • G • , • • (4) • • (2),
• • • . F• • • • ,
$\mathrm{f}_i^* = [(m_i^*) \;, vec[\dot{\mathrm{S}}_i^*] \;] \;,$ $\dot{\mathrm{S}}_i^* = -\frac{1}{2}(\mathrm{S}_i^* + m_i^*(m_i^*) \;),$ $\hat{\mathrm{f}}_i^* = [(m_i^*) \;, vec[\mathrm{S}_i^*] \;] \;.$

Lemma 5. *Suppose that* $P_i(x|\mathrm{f}_i^*) = P_i(x|m_i^*, \mathrm{S}_i^*)$ *is a Gaussian density and* $\mathrm{k}(\mathrm{S}_i^*)$ *is upper bounded. As* 1^i *tends to zero, we have*

$$\|I(\mathrm{f}_i^*)\| = O((1^i \quad)^{-\mathrm{t}}), \tag{30}$$

where t *is a positive number.*

Proof. B ● ,

$$\frac{\partial P_i(x|m_i^*, \Sigma_i^*)}{\partial m_i^*} = (x - m_i^*)\Sigma_i^* P_i(x|m_i^*, \Sigma_i^*), \tag{31}$$

$$\frac{\partial P_i(x|m_i^*, \Sigma_i^*)}{\partial \Sigma_i^*} = -\frac{1}{2}(\Sigma_i^{*-1} - \Sigma_i^{*-1}(x - m_i^*)(x - m_i^*) \;\; \Sigma_i^{*-1})P_i(x|m_i^*, \Sigma_i^*). \tag{32}$$

A ● ● ● ● F ● ● , ●

$$I(\phi_i^*) = E_{\phi_i^*}\left(\left(\frac{\partial P_i(X|\phi_i^*)}{\partial \phi_i^*}\right)\left(\frac{\partial P_i(X|\phi_i^*)}{\partial \phi_i^*}\right) \;\right)$$
$$= E_{\phi_i^*}\left(\frac{\partial(\hat{\phi}_i^*)}{\partial \phi_i^*}\left(\frac{\partial P_i(X|\phi_i^*)}{\partial \hat{\phi}_i^*}\right)\left(\frac{\partial P_i(X|\phi_i^*)}{\partial \hat{\phi}_i^*}\right) \;\left(\frac{\partial(\hat{\phi}_i^*)}{\partial \phi_i^*}\right) \;\right)$$
$$= \frac{\partial(\hat{\phi}_i^*)}{\partial \phi_i^*} I(\hat{\phi}_i^*)\left(\frac{\partial(\hat{\phi}_i^*)}{\partial \phi_i^*}\right) \;,$$

$$I(\hat{\phi}_i^*) = E_{\phi_i^*}\left(\left(\frac{\partial P_i(X|\phi_i^*)}{\partial \hat{\phi}_i^*}\right)\left(\frac{\partial P_i(X|\phi_i^*)}{\partial \hat{\phi}_i^*}\right) \;\right).$$

I E . (31) (32) ● ● ● ●
$P_i^3(x|m_i^*, \Sigma_i^*)$ ● $I(\hat{\phi}_i^*)$ ● ● G
$P_i(x|m_i^*, \frac{1}{3}\Sigma_i^*)$ ● ● ● $|\Sigma_i^*|$ ● ●
● ,

$$I(\hat{\phi}_i^*) = E_{(m_i^*,(1/3)\Sigma_i^*)}(G(X, \phi_i^*)),$$

$G(x, \phi_i^*)$ ● ● $x - m_i^*$ Σ_i^*. ● ●
$y = x - m_i^*$,

$$I(\hat{\phi}_i^*) = E_{(0,(1/3)\Sigma_i^*)}(G(Y, \Sigma_i^*)),$$

$G(y, \Sigma_i^*)$ ● ● ● $g_{pq}(y, \Sigma_i^*)$ ● ● ●
● $y_1, \ldots, y_n$. I Σ_i^{*-1} ●

$$\Sigma_i^{*-1} = |\Sigma_i^*|^{-1}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d_i} \\ a_{21} & a_{22} & \cdots & a_{2d_i} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d_i1} & a_{d_i2} & \cdots & a_{d_id_i} \end{pmatrix},$$

● a_{kl} ● ● ● σ_{kl}^{*i} Σ_i^*,
● $g_{pq}(y, \Sigma_i^*)$ ● ● ● ● ● σ_{kl}^{*j}
● ● ● $|\Sigma_i^*|$. σ_{kl}^{*i}

• , • t • •
• $(1^{i}\quad)^{t} g_{pq}(y, \mathrm{S}_i^*)$ • . B • •
G • 1 $< B$, $E_{(0,1/3\mathrm{S}_i^*)}((1^{i}\quad)^{t} g_{pq}(Y, \mathrm{S}_i^*))$ •
• . • ,

$$\begin{aligned}\|I(\hat{\mathrm{f}}_i^*)\| &= \|(1^{i}\quad)^{-t}(1^{i}\quad)^{t} I(\mathrm{f}_i^*)\| \\ &= (1^{i}\quad)^{-t}\|(1^{i}\quad)^{t} I(\mathrm{f}_i^*)\| \\ &\leqslant \mathrm{o}(1^{i}\quad)^{-t},\end{aligned}$$

o • . B

$$\|I(\mathrm{f}_i^*)\| \leqslant \left\|\frac{\partial(\hat{\mathrm{f}}_i^*)}{\partial \mathrm{f}_i^*}\right\| \|I(\hat{\mathrm{f}}_i^*)\| \left\|\left(\frac{\partial(\hat{\mathrm{f}}_i^*)}{\partial \mathrm{f}_i^*}\right)\quad\right\| = 4\|I(\hat{\mathrm{f}}_i^*)\|,$$

$\|\partial(\hat{\mathrm{f}}_i^*)\ /\partial \mathrm{f}_i^*\| = \|(\partial(\hat{\mathrm{f}}_i^*)\ /\partial \mathrm{f}_i^*)\ \| = 2$, •
• • , E . (30) • . □

• , • • • • (1) (3)
• • • • EM • • G
, ,

Theorem 2. *Given a Gaussian mixture of K densities of the parameter* F^* *that satisfies conditions* (1) (3), *as* $e(\mathrm{F}^*)$ *tends to zero as an infinitesimal, we have*

$$\|G'(\mathrm{F}^*)\| = o(\ ^{0.5-\mathrm{e}}(\mathrm{F}^*)), \tag{33}$$

where e *is an arbitrarily small positive number*.

I • • , • 1 • G • • •
(1) (3) .

5. Conclusions

I • • • ,
• • • • • •
• • , • • • • EM •
• -N • • . M• • , •
• • • •
•z •.

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• • G • G C• •
H• K• A (P • •. C HK4225/04E) G • N
F• • • C (P • •. 60071004).

Appendix

Proof of Lemma 1. • • ( ). F• • • , $Z(F^*) = \quad {}_{i \neq j} Z_i(m_j^*) = Z_{i'}(m_{j'}^*)$. A • • • • (2) (3), • • $a_1, a_2, b_1, b_2, c_1, c_2$

$$a_1(1^{i'} \quad)^n \leqslant (1^{i} \quad)^n \leqslant a_2(1^{i'} \quad)^n, \tag{34}$$

$$b_1(1^{i} \quad)^n \leqslant (1^{j} \quad)^n \leqslant b_2(1^{i} \quad)^n, \tag{35}$$

$$c_1 \| m_{i'}^* - m_{j'}^* \| \leqslant \| m_i^* - m_j^* \| \leqslant c_2 \| m_{i'}^* - m_{j'}^* \|. \tag{36}$$

C• E . (34) (35) E . (36), • • • • a_1', a_2', b_1', b_2'

$$a_1' Z(F^*) \leqslant Z_i(m_j^*) \leqslant a_2' Z(F^*),$$

$$b_1' Z_i(m_j^*) \leqslant Z_j(m_i^*) \leqslant b_2' Z_i(m_j^*).$$

• , $Z(F^*), Z_i(m_j^*)$ $Z_j(m_i^*)$. • ( ), $\| m_i^* - m_j^* \|$ • $Z(F^*) \to 0$, E . (23) • • • . I $\| m_i^* - m_j^* \|$ • $Z(F^*) \to 0$, $\| m_i^* \| \leqslant \| m_i^* - m_j^* \| + \| m_j^* \|$, E . (23) • $\| m_i^* \|$ • $\| m_j^* \|$ • . O , • $\| m_i^* \|$ $\| m_j^* \|$ • , • • $\| m_i^* \|$ • • • • $\| m_i^* - m_j^* \|$, • • E . (23). • , ( ) • • .

F , • ( ) • • • .

I $p = q > 0$, • • ( ),

$$\| m_i^* - m_j^* \|^p (1^{i} \quad)^{-nq} = \| m_i^* - m_j^* \|^p (1^{i} \quad)^{-np}$$
$$= (Z_i(m_j^*))^{-p} = O(Z^{-p}(F^*)) = O(Z^{-p \vee q}(F^*)).$$

I $p > q$, 1^i • • • ( ),

$$\| m_i^* - m_j^* \|^p (1^{i} \quad)^{-nq} \leqslant O(Z^{-p}(F^*)) = O(Z^{-p \vee q}(F^*)).$$

I $p < q$, $\| m_i^* - m_j^* \| \geqslant T$,

$$\| m_i^* - m_j^* \|^p (1^{i} \quad)^{-nq} \leqslant O(Z^{-q}(F^*)) = O(Z^{-p \vee q}(F^*)).$$

• ,

$$\| m_i^* - m_j^* \|^p (1^{i} \quad)^{-nq} \leqslant O(Z^{-p \vee q}(F^*)). \qquad \square$$

Proof of Lemma 2. • • ( ). A • • • • ,

$$\| P_i \| = \| E_{f_i^*}((t_i(X) - f_i^*)(t_i(X) - f_i^*) \quad) \| \leqslant E_{f_i^*}(\| t_i(X) - f_i^* \|^2). \tag{37}$$

$t_i(x)$ • • • • $x_1, x_2, \ldots, x_n$, . ., • • • x, • • • • • :

$$t_i(x) = P_0 + P_1 x + P_2 x^2 + \cdots + P_k x^k,$$

$k \geqslant 0$, P_i $d_i \times n^i$, x^i ● ● ●
● $x_{j_1} x_{j_2} \cdots x_{j_i}$ ● ● , x_{j_p} ● ●
$x_1, x_2, \ldots, x_n$. I $\|x^i\| \leqslant \sqrt{n}\|x\|^i$ ● $i = 0, 1, \ldots, k$.
B ● ● ● ,

$$t_i(x) = t_i(x - m_i^* + m_i^*)$$
$$= P'_0 + P'_1(x - m_i^*) + P'_2(x - m_i^*)^2 + \cdots + P'_k(x - m_i^*)^k, \quad (38)$$

P'_i $d_i \times n^i$, ● ● ● $m_{i1}^*, \ldots, m_{in}^*$.

$$\text{f}_i^* = E_{\text{f}_i^*}(t_i(X)) = P'_0 + E_{\text{f}_i^*}(P'_1(X - m_i^*)) + \cdots + E_{\text{f}_i^*}(P'_k(X - m^*)^k) \quad (39)$$

$$E_{\text{f}_i^*}(P'_1(X - m_i^*)) = P'_1 E_{\text{f}_i^*}(X - m_i^*) = 0,$$

$$t_i(X) - \text{f}_i^* = \sum_{j=1}^{k} [P'_j(X - m_i^*)^j - E_{\text{f}_i^*}(P'_j(X - m_i^*)^j)].$$

N● ,

$$E_{\text{f}_i^*}(\|t_i(X) - \text{f}_i^*\|^2) = E_{\text{f}_i^*}(\|(t_i(X) - \text{f}_i^*)\ (t_i(X) - \text{f}_i^*)\|)$$
$$= E_{\text{f}_i^*}\left(\left\| \sum_{j_1=1, j_2=1}^{k} [P'_{j_1}(X - m_i^*)^{j_1} - E_{\text{f}_i^*}(P'_{j_1}(X - m_i^*)^{j_1})] \times [P'_{j_2}(X - m_i^*)^{j_2} - E_{\text{f}_i^*}(P'_{j_2}(X - m_i^*)^{j_2})] \right\|\right)$$
$$\leqslant \sum_{j_1=1, j_2=1}^{k} E_{\text{f}_i^*}(\|[P'_{j_1}(X - m_i^*)^{j_1} - E(P'_{j_1}(X - m_i^*)^{j_1})]\ \| \times \|[P'_{j_2}(X - m_i^*)^{j_2} - E_{\text{f}_i^*}(P'_{j_2}(X - m_i^*)^{j_2})]\|)$$
$$\leqslant \sum_{j_1=1, j_2=1}^{k} E_{\text{f}_i^*}^{1/2}(\|P'_{j_1}(X - m_i^*)^{j_1} - E_{\text{f}_i^*}(P'_{j_1}(X - m_i^*)^{j_1})\|^2) \times E_{\text{f}_i^*}^{1/2}\|P'_{j_2}(X - m_i^*)^{j_2} - E_{\text{f}_i^*}(P'_{j_2}(X - m_i^*)^{j_2} \,|\, \text{f}_i^*)\|^2). \quad (40)$$

P ,

$$E_{\text{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1} - E_{\text{f}_i^*}(P'_{j_1}(X - m_i^*)^{j_1})\|^2)$$
$$= E_{\text{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1}\|^2) - \|E_{\text{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1})\|^2$$
$$\leqslant E_{\text{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1}\|^2) \leqslant \sqrt{n} E_{\text{f}_i^*}(\|P'_{j_1}\|^2 \|X - m_i^*\|^{2j_1})$$
$$= \sqrt{n}\|P'_{j_1}\|^2 E_{\text{f}_i^*}(\|X - m_i^*\|^{2j_1}). \quad (41)$$

B $P_i(x|\,\mathrm{f}_i^*) \leqslant U_i(x|\,\mathrm{f}_i^*)$,

$$E_{\mathrm{f}_i^*}(\|X - m_i^*\|^{2j_1}) \leqslant \int \|x - m_i^*\|^{2j_1} U_i(x|\,\mathrm{f}_i^*) \quad x$$
$$= \int \|y\|^{2j_1} w(y + m_i^*)(1^i \quad)^{-c_1} \quad {}^{-\mathrm{r}(1/(1^i \quad)^{\mathrm{n}c_2})\|y\|^{c_2}} \quad y, \tag{42}$$

• • $y = x - m_i^*$. $w(x)$ • • • ,

$$w(y + m_i^*) \leqslant w_0 + w_1\|y\| + \cdots + w_{k'}\|y\|^{k'}, \tag{43}$$

k' • , $w_0, w_1, \ldots, w_{k'}$ • • • • •
• $\|m_i^*\|$, . .,

$$w_i = w_0^i + w_1^i\|m_i^*\| + \cdots + w_{c_i}^i\|m_i^*\|^{c_i} \quad \bullet \quad i = 0, 1, \ldots, k', \tag{44}$$

$w_0^i, w_1^i, \ldots, w_{c_i}^i$ • , $c_0, \ldots, c_{k'}$ •
. B L 1,

$$w_i \leqslant v_0^i + v_1^i\|m_i^* - m_j^*\| + \cdots + v_{c_i}^i\|m_i^* - m_j^*\|^{c_i} \quad \bullet \quad i = 0, 1, \ldots, k', \tag{45}$$

$v_0^i, v_1^i, \ldots, v_{c_i}^i$ • . • • $w(y + m_i^*)$
• E . (42),

$$E_{\mathrm{f}_i^*}(\|X - m_i^*\|^{2j_1}) \leqslant \sum_{l=0}^{k'} w_l(1^i \quad)^{-c_1} \int \|y\|^{2j_1+l} \quad {}^{-\mathrm{r}(1/(1^i \quad)^{\mathrm{n}c_2})\|y\|^{c_2}} \quad y$$
$$= \sum_{l=0}^{k'} w_l(1^i \quad)^{-c_1+\mathrm{n}(2j_1+l+1)} \int \|u\|^{2j_1+l} \quad {}^{-\mathrm{r}\|u\|^{c_2}} \quad u,$$

• • $u = y/(1^i \quad)^{\mathrm{n}}$. C , $\int \|u\|^{2j_1+l} \quad {}^{-\mathrm{r}\|u\|^{c_2}} \quad u$
• j_1 • . 1^i • ,
$E_{\mathrm{f}_i^*}(\|X - m_i^*\|^{2j_1})$ • • • • • $\|m_i^* - m_j^*\|$.
M• • , • P'_{j_1} • • • $m_{i1}^*, \ldots, m_{in}^*$, $\|P'_{j_1}\|$
• • • • • $\|m_i^*\|$. • , $E_{\mathrm{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1}\|^2)$
• • • • • $\|m_i^* - m_j^*\|$.
A , $E_{\mathrm{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1} - E_{\mathrm{f}_i^*}(P'_{j_1}(X - m_i^*)^{j_1})\|^2)$ •
• • • • $\|m_i^* - m_j^*\|$. $\|m_i^* - m_j^*\| \geqslant T'$,

$$E_{\mathrm{f}_i^*}(\|P'_{j_1}(X - m_i^*)^{j_1} - E_{\mathrm{f}_i^*}(P'_{j_1}(X - m_i^*)^{j_1})\|^2) \leqslant \mathrm{c}_{j_1}\|m_i^* - m_j^*\|^{p_{j_1}}, \tag{46}$$

c_{j_1} p_{j_1} • • .
E . (46) • E . (40),

$$E_{\mathrm{f}_i^*}(\|t_i(X) - \mathrm{f}_i^*\|^2) \leqslant \mathrm{c}\|m_i^* - m_j^*\|^p, \tag{47}$$

c p • . • , • • E . (37), ( ) •
• .

A ● ( ), $j \neq i$, $\mathrm{f}'_j = E_{\mathrm{f}^*_j}(t_i(X))$

$$\begin{aligned} E_{\mathrm{f}^*_j}(\|t_i(X) - \mathrm{f}^*_i\|^2) &\leqslant E_{\mathrm{f}^*_j}((\|t_i(X) - \mathrm{f}'_j\| + \|\mathrm{f}'_j - \mathrm{f}^*_i\|)^2) \\ &= E_{\mathrm{f}^*_j}(\|t_i(X) - \mathrm{f}'_j\|^2 + 2\|t_i(X) - \mathrm{f}'_j\|\|\mathrm{f}'_j - \mathrm{f}^*_i\| + \|\mathrm{f}'_j - \mathrm{f}^*_i\|^2) \\ &\leqslant E_{\mathrm{f}^*_j}(2\|t_i(X) - \mathrm{f}'_j\|^2 + 2\|\mathrm{f}'_j - \mathrm{f}^*_i\|^2) \\ &= 2E_{\mathrm{f}^*_j}(\|t_i(X) - \mathrm{f}'_j\|^2) + 2\|\mathrm{f}^*_i - \mathrm{f}'_j\|^2. \end{aligned} \tag{48}$$

I ● , ●

$$E_{\mathrm{f}^*_j}(\|t_i(X) - \mathrm{f}'_j\|^2) \leqslant \mathrm{c}_1\|m^*_i - m^*_j\|^{p_1}, \tag{49}$$

c_1 p_1 ● . M● ● ,

$$\|\mathrm{f}^*_i - \mathrm{f}'_j\| \leqslant \|\mathrm{f}^*_i\| + \|\mathrm{f}'_j\|.$$

B E . (38), ● ● $\|\mathrm{f}^*_i\|$ $\|\mathrm{f}'_j\|$ ● $\mathrm{c}_2\|m^*_i - m^*_j\|^{p_2}$, c_2 p_2 ● ● . ● , E . (48), $E_{\mathrm{f}^*_j}(\|t_i(X) - \mathrm{f}^*_i\|^2)$ ● ● ● ● ● $\|m^*_i - m^*_j\|$. $\|m^*_i - m^*_j\| \geqslant T'$,

$$E_{\mathrm{f}^*_j}(\|t_i(X) - \mathrm{f}^*_i\|^2) \leqslant \mathrm{c}_j\|m^*_i - m^*_j\|^{p_j}, \quad j \neq i, \tag{50}$$

c_j p_j ● .

B E . (47) (50),

$$E(\|t_i(X) - \mathrm{f}^*_i\|^2) = \sum_{j=1}^{K} \mathrm{a}^*_j E_{\mathrm{f}^*_j}(\|t_i(X) - \mathrm{f}^*_i\|^2) \leqslant \mathrm{u}M^q_i(\mathrm{F}^*),$$

$M_i(\mathrm{F}^*) = \quad {}_{j \neq i}\|m^*_i - m^*_j\|$, u q ● . □

Proof of Lemma 3. ●

$$f(\mathrm{Z}) = o(\mathrm{Z}^p),$$

$\mathrm{Z} \to 0$, p ● .

● ● K ● ● ● F^* ● $\mathrm{Z}(\mathrm{F}^*) = \mathrm{Z}$. $i \neq j$, ● ● Z, ● m^*_{ij} ● m^*_i m^*_j

$$\mathrm{a}^*_i P_i(m^*_{ij}|\,\mathrm{f}^*_i) = \mathrm{a}^*_j P_j(m^*_{ij}|\,\mathrm{f}^*_j).$$

$$E_i = \{x : \mathrm{a}^*_i P_i(x|\,\mathrm{f}^*_i) \geqslant \mathrm{a}^*_j P_j(x|\,\mathrm{f}^*_j)\},$$

$$E_j = \{x : \mathrm{a}^*_j P_j(x|\,\mathrm{f}^*_j) > \mathrm{a}^*_i P_i(x|\,\mathrm{f}^*_i)\}.$$

A $\mathrm{Z}(\mathrm{F}^*)$ ● z ●, $(1^i\ )^{\mathrm{n}}/(\|m^*_i - m^*_j\|)$ $(1^j\ )^{\mathrm{n}}/(\|m^*_i - m^*_j\|)$ ● . M● ● , $\mathrm{k}(\mathrm{S}^*_i)$ $\mathrm{k}(\mathrm{S}^*_j)$ ● ● . , ● ●● (. .,) ● m^*_i (● m^*_j) E_i (● E_j). F● , $\mathcal{N}_{r_i}(m^*_i)$ $\mathcal{N}_{r_j}(m^*_j)$ ● ●● E_i E_j, , r_i r_j . $\mathrm{k}(\mathrm{S}^*_i)$ $\mathrm{k}(\mathrm{S}^*_j)$

● , r_i r_j ● ● ● ● ● $\|m_i^* - m_j^*\|$ $\|m_i^* - m_j^*\|$
● ● ● . ● ● ● b_1
b_2

$$r_i \geqslant b_i \|m_i^* - m_j^*\| \qquad r_j \geqslant b_j \|m_i^* - m_j^*\|.$$

$$\mathscr{D}_i = \mathscr{N}^c_{r_i}(m_i^*) = \{x\colon \|x - m_i^*\| \geqslant r_i\},$$
$$\mathscr{D}_j = \mathscr{N}^c_{r_j}(m_j^*) = \{x\colon \|x - m_j^*\| \geqslant r_j\}$$

$$E_i \subset D_j, \qquad E_j \subset D_i.$$

M● ● , ● ● ● $e_{ij}(\mathrm{F}^*)$ $h_k(x)$

$$\begin{aligned} e_{ij}(\mathrm{F}^*) &= \int h_i(x)h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} \\ &= \int_{E_i} h_i(x)h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} + \int_{E_j} h_i(x)h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} \\ &\leqslant \int_{\mathscr{D}_j} h_i(x)h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} + \int_{\mathscr{D}_i} h_i(x)h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} \\ &\leqslant \int_{\mathscr{D}_j} h_j(x)P(x|\mathrm{F}^*)\ \mathrm{m} + \int_{\mathscr{D}_i} h_i(x)P(x|\mathrm{F}^*)\ \mathrm{m} \\ &= \mathrm{a}_j^* \int_{\mathscr{D}_j} P_j(x|\,\mathrm{f}_j^*)\ \mathrm{m} + \mathrm{a}_i^* \int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*)\ \mathrm{m}. \end{aligned}$$

● ● $\int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*)\ \mathrm{m}.$ $r_i \geqslant b_i \|m_i^* - m_j^*\|,$

$$\int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*)\ \mathrm{m} \leqslant \int_{\|x - m_i^*\| \leqslant b_i \|m_i^* - m_j^*\|} P_i(x|\,\mathrm{f}_i^*)\ \mathrm{m}.$$

B ● ● $y = (x - m_i^*)/\|m_i^* - m_j^*\|,$

$$\begin{aligned} &\int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*)\ \mathrm{m} \\ &\quad\leqslant \int_{\|y\| \leqslant b_i} w(\|m_i^* - m_j^*\|y + m_i^*)(1^i \quad)^{-c_1} \ {}^{-\mathrm{r}(\|m_i^* - m_j^*\|^{c_2})/(1 \quad)^{\mathrm{n}c_2}\|y\|^{c_2}} \|m_i^* - m_j^*\|\ \mathrm{m}' \\ &\quad= \int_{\|y\| \leqslant b_i} \|m_i^* - m_j^*\| w(\|m_i^* - m_j^*\|y + m_i^*)(1^i \quad)^{-c_1} \\ &\qquad\times \ {}^{-\mathrm{r}(\|m_i^* - m_j^*\|^{c_2})/(1 \quad)^{\mathrm{n}c_2}\|y\|^{c_2}}\ \mathrm{m}', \end{aligned} \tag{51}$$

m' ● ● m ● ● .
● ● ● ● $w(\|m_i^* - m_j^*\|y + m_i^*)$
● ● ● ● m_i^* ● ● ● ● $\|m_i^* - m_j^*\|,$
● q ● ● ●
● $\|m_i^* - m_j^*\|^{-q} w(\|m_i^* - m_j^*\|y + m_i^*)$ ● $\mathrm{Z}(\mathrm{F}^*) \to 0.$ ●

$\|m_i^* - m_j^*\|^{-q} w(\|m_i^* - m_j^*\| y + m_i^*)$ ● ● ●
● ● ● ● y ● ● . M● ● , L 1,

$$\|m_i^* - m_j^*\|^{1+q}(1^{i} \quad)^{-c_1} \leqslant O(\mathrm{Z}^{-c_1'}),$$
$$\|m_i^* - m_j^*\|^{c_2}(1^{i} \quad)^{-\mathrm{n}c_2} \geqslant O(\mathrm{Z}^{-c_2}),$$

$c_1' = (q+1) \vee (c_1/\mathrm{n})$.

A ● ● , ● E . (51)

$$\int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*) \quad \mathrm{m} \leqslant \int_{\mathscr{B}_i} \frac{1}{\mathrm{Z}^{c_1'}(\mathrm{F}^*)} w_1(y) \quad {}^{-\mathrm{r}'(1/\mathrm{Z}^{c_2}(\mathrm{F}^*))\|y\|^{c_2}} \quad \mathrm{m}'$$
$$= \int_{\mathscr{B}_i} \frac{1}{\mathrm{Z}^{c_1'}} w_1(y) \quad {}^{-\mathrm{r}'(1/\mathrm{Z}^{c_2})\|y\|^{c_2}} \quad \mathrm{m}', \qquad (52)$$

$\mathscr{B}_i = \{y : \|y\| \geqslant b_i\}$, r′ ● ● , $w_1(y)$ ●
● ● ● ● y ● ● .

F ● ,

$$F_i(\mathrm{Z}) = \int_{\mathscr{B}_i} P(y|\mathrm{Z}) \quad y, \quad P(y|\mathrm{Z}) = \frac{1}{\mathrm{Z}^{c_1'}} w_1(y) \quad {}^{-\mathrm{r}'(1/\mathrm{Z}^{c_2})\|y\|^{c_2}}$$

● ● $F_i(\mathrm{Z})/\mathrm{Z}^p$ Z ●z ●.

F● $y \in \mathscr{B}_i$,

$$\underset{\mathrm{Z}\to 0}{} \frac{P(y|\mathrm{Z})}{\mathrm{Z}^p} = w_1(y) \underset{\mathrm{Z}\to 0}{} \frac{1}{\mathrm{Z}^{(c_1'+p)}} \quad {}^{-\mathrm{r}'(1/\mathrm{Z}^{c_2})\|y\|^{c_2}}$$
$$= w_1(y) \underset{\mathrm{z}=\frac{1}{\mathrm{Z}}\to\infty}{} \frac{\mathrm{z}^{(c_1'+p)}}{\mathrm{z}^{c_2}\,\mathrm{r}'\|y\|^{c_2}}$$
$$= 0,$$

● $\mathscr{B}_i$, ●

$$\underset{\mathrm{Z}\to 0}{} \frac{F_i(\mathrm{Z})}{\mathrm{Z}^p} = \underset{\mathrm{Z}\to 0}{} \int_{\mathscr{B}_i} \frac{P(y|\mathrm{Z})}{\mathrm{Z}^p} \quad \mathrm{m}'$$
$$= \int_{\mathscr{B}_i} \underset{\mathrm{Z}\to 0}{} \frac{P(y|\mathrm{Z})}{\mathrm{Z}^p} \quad \mathrm{m}'$$
$$= 0$$

$F_i(\mathrm{Z}) = o(\mathrm{Z}^p)$. I ● ● ● E . (52)

$$\underset{\mathrm{Z}(\mathrm{F}^*)=\mathrm{Z}}{} \int_{\mathscr{D}_i} P_i(x|\,\mathrm{f}_i^*) \quad \mathrm{m} = o(\mathrm{Z}^p). \qquad (53)$$

, ● ● :

$$\underset{\mathrm{Z}(\mathrm{F}^*)=\mathrm{Z}}{} \int_{\mathscr{D}_j} P_j(x|\,\mathrm{f}_j^*) \quad \mathrm{m} = o(\mathrm{Z}^p).$$

A ,

$$f_{ij}(Z) = \sum_{Z(F^*)=Z} e_{ij}(F^*)$$

$$\leqslant \sum_{Z(F^*)=Z} \left(a_j^* \int_{\mathscr{D}_j} P_j(x|f_j^*) \; m + a_i^* \int_{\mathscr{D}_i} P_i(x|f_i^*) \; m \right)$$

$$\leqslant \sum_{Z(F^*)=Z} \int_{\mathscr{D}_j} P_j(x|f_j^*) \; x + \sum_{Z(F^*)=Z} \int_{\mathscr{D}_i} P_i(x|f_i^*) \; m)$$

$$= o(Z^p).$$

,

$$f(Z) \leqslant \sum_{ij} f_{ij}(Z) = o(Z^p). \tag{54}$$

M● ● ,

$$\lim_{Z\to 0} \frac{f^e(Z)}{Z^p} = \lim_{Z\to 0} \left(\frac{f(Z)}{Z^{\frac{p}{e}}}\right)^e = 0,$$

$f^e(Z) = o(Z^p)$ $f^e(Z(F^*)) = o(Z^p(F^*))$. □

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Jinwen Ma M ● ● ' J ●● 1988 P .D. ● ● ● N 1992. F ● J 1992 ● N● 1999, L ● A ● ● ● D ● M , ● . F ● D 1999, ● ● ● I ● M , ● . I 2001, ● D ● I ● ● ●● ● M , P . D 1995 2004, ● D ● C● & E , C ● H● K● A ● ● F ● . H ● 50 ● ● , ● ● , , ● ● ● .

Lei Xu (IEEE F ●) ● ● ● C● & E , C ● H● K● . H ● P .D. ● 1986, ● D ● M , P 1987 ● ● ● ● ● ● ● ● ● 1988. D 1989 93, ● F , C A, H MI . H ● C HK 1993 ● , ● ● 1996 ●● ● ● 2002. P ● . ● ● ● ● ● , N N ● , IEEE ● ● N N ● , ● ● ● I ● N N ● ● (01-03), ● C● ● F C● ● IEEE C● ● I ● (01-03), ● A -P N N ● A . P ● . 100 ● ● , ● ● , ●● , ● ● , ● . H ● ● / / / ● ● ● ● , ● ● ● ● ● ● ● . H C ● ● (1994 C N ● N A) ● (1995 INN L A). P ● . IEEE F ● F ● ● I ● A ● ● ● P ● ● , ● E ● A ● .