Contrast Functions for Non-circular and Circular Sources Separation in Complex-Valued ICA

***Abstract*— In this paper, the complex-valued ICA problem is studied in the context of blind complex-source separation. We formulate the complex ICA problem in a general setting, and define the superadditive functional that may be used for constructing a contrast function for circular complex sources separation. We propose several contrast functions and study their properties. Finally, we also discuss relevant issues and present the convex analysis of a specific contrast function.**

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- - ( ):

$$\mathbb{E}[\ ] = \mathbb{E}[\quad + \quad] = \mathbb{E}[\quad] + \ \mathbb{E}[\quad].$$

- - :

$$\mathbb{E}[\ ^2] = \mathbb{E}[\ ^2\ ] - \mathbb{E}[\ ^2\ ] + 2\ \mathbb{E}[\quad].$$

- - ( ):

$$\mathtt{var}[\ ] = \mathbb{E}[|\ - \mathbb{E}[\ ]|^2] = \mathbb{E}[|\ |^2] - \big|\mathbb{E}[\ ]\big|^2.$$

i j $C_{ij} = \mathbb{E}\big[(\ _i - \mathbb{E}[\ _i])(\ _j^* - \mathbb{E}[\ _j^*])\big] = \mathbb{E}[\ _i\ _j^*] - \mathbb{E}[\ _i]\mathbb{E}[\ _j^*]$. $_i$ $_j$ ($\neq i$) *uncorrelated* $C_{ij} = 0$.

$\mathbf{z} = [\ _1, \ldots, \ _n]$ - , $\mathbf{z}^H = [\ _1^*, \ldots, \ _n^*]^T \equiv (\mathbf{z}^*)^T$ H (. .,), $\mathbb{E}[\mathbf{z}]$, $\mathtt{cov}[\mathbf{z}] \equiv \mathbb{E}\big[(\mathbf{z} - \mathbb{E}[\mathbf{z}])(\mathbf{z} - \mathbb{E}[\mathbf{z}])^H\big]$; , *pseudo-covariance matrix* $\mathtt{pcov}[\mathbf{z}] \equiv \mathbb{E}\big[(\mathbf{z} - \mathbb{E}[\mathbf{z}])(\mathbf{z} - \mathbb{E}[\mathbf{z}])^T\big]$.

Definition 1: A - α, ( ) ($e^{j\alpha}$) (. ., ( )).

- $\mathbf{z}$ *second-order circular* - (. .,); $\mathbb{E}[\ ] = 0$, - $\mathbb{E}[\ ^2] = 0$

. $\mathbb{E}[\mathbf{z}\mathbf{z}^H]$, $\mathbf{z}$; $\mathbf{z}$ *strongly uncorrelated* $\mathbb{E}[\mathbf{z}\mathbf{z}^H]$ $\mathbb{E}[\mathbf{z}\mathbf{z}^T]$. $\mathbf{z}$, $\mathbf{z}$ *symmetric*; $\mathbf{z}$, $\mathbf{z}$ - , $\mathbb{E}[\mathbf{z}\mathbf{z}^T] = \mathbf{0}$.

, k k *zero-mean* 1 :

$$\mathtt{skewness}(\ ) = \mathbb{E}[|\ |^3]/\big(\mathbb{E}[|\ |^2]\big)^{3/2}, \tag{1}$$

$$\mathtt{kurtosis}(\ ) = \mathbb{E}[|\ |^4] - 2\big(\mathbb{E}[|\ |^2]\big)^2 - \big|\mathbb{E}[\ ^2]\big|^2. \tag{2}$$

Definition 2: $_1$ $_2$ $(\ _1, \ _2) = \ (\ _1)\ (\ _2)$; $_1$ $_2$, $(|\ _1|, |\ _2|) = \ (|\ _1|)\ (|\ _2|)$.

$J(\mathbf{z})$ - , $\mathbf{z} \in \mathbb{C}^N$, :

$$\nabla J \equiv \frac{\partial J(\mathbf{z})}{\partial \mathbf{z}^*} = \frac{1}{2}\Big(\frac{\partial J(\mathbf{z})}{\partial \mathbf{z}} + \ \frac{\partial J(\mathbf{z})}{\partial \mathbf{z}}\Big) = 0.$$

, $\frac{\partial J(\mathbf{z})}{\partial \mathbf{x}} = \frac{\partial J(\mathbf{z})}{\partial \mathbf{z}} = 0$.

Definition 3: A - $J(\mathbf{z})$ ($\mathbf{z} \in \mathbb{C}^N$)

$$J\big(\lambda\mathbf{z}_1 + (1-\lambda)\mathbf{z}_2\big) \le \lambda J(\mathbf{z}_1) + (1-\lambda)J(\mathbf{z}_2)$$

$\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{C}^N$ $0 \le \lambda \le 1$. A , H , $\mathbf{H} = \frac{\partial^2 J(\mathbf{z})}{\partial \mathbf{z}\partial \mathbf{z}^H}$, (. .,), $J(\mathbf{z})$.

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- A, - A : $\mathbf{x} = \mathbf{A}\mathbf{s}$, $\mathbf{s} \in \mathbb{C}^n$ - - , - , $\mathbf{x} \in \mathbb{C}^n$ - - , $\mathbf{A} \in \mathbb{C}^{n\times n}$ (. ., k) - . - A:

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- ;
- .

- A; - . , , . , k : ( ) ( ) , (A; ( ) - , . , ; A 20 .

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$\mathbb{E}[\mathbf{x}] = 0$ $\mathbb{E}[\mathbf{x}\mathbf{x}^H] = \mathbf{I}$,

k

strong-uncorrelating transform 10, 11.

k *mutual information*, k-

:

$$I(\ _1, \ldots, \ _n) = \mathbb{E}_{p(\mathbf{y})}\Big[\log \frac{(\mathbf{y})}{\prod_{i\ 1}^{n}\ (\ _i)}\Big] = \int (\mathbf{y}) \log \frac{(\mathbf{y})}{\prod_{i\ 1}^{n}\ (\ _i)} d\mathbf{y} \quad (4)$$

(4) $\{\ _1, \ldots, \ _n\}$ $\{\ _1, \cdots, \ _n\}$,

(4)

$$I(\ _1, \ldots, \ _n) = -\frac{1}{2}\log\Big(\frac{(\mathbf{C_y})}{\prod_{i\ 1}^{n} C_{ii}}\Big) = -\frac{1}{2}\sum_{i\ 1}^{n}\log(\lambda_i) \quad (5)$$

$C_{ij} = \mathbb{E}[(\ _i - \mathbb{E}[\ _i])(\ _j^* - \mathbb{E}[\ _j^*])]$, λ_i

: $\mathbf{C_y u} = \lambda \mathbf{\Lambda u}$, $\mathbf{C_y} = \mathrm{cov}[\mathbf{y}]$, $\mathbf{\Lambda} = \{C_{11}, C_{22}, \ldots, C_{nn}\}$.

$$I(\ _1, \ldots, \ _n) = \sum_{i\ 1}^{n} H(\ _i) - H(\mathbf{y}),$$

$H(\mathbf{y})$ - $\{\ _1, \ldots, \ _n\}$; $H(\mathbf{y}) = H(\mathbf{x}) + \log|\ (\mathbf{W})|$, (4)

$$J(\mathbf{W}) = \sum_{i\ 1}^{n} H(\ _i) - \log|\ (\mathbf{W})| - H(\mathbf{x}), \quad (6)$$

$H(\mathbf{x})$

(. . .) $\mathbf{W}$. k (6) . . . $\mathbf{W}^*$

$$\nabla_{\mathbf{W}^*} J(\mathbf{W}) = \Big(\mathbb{E}_{\mathbf{y}}[\psi(\mathbf{y})\mathbf{y}^H] - \mathbf{I}\Big)\mathbf{W}^{-H}, \quad (7)$$

$\mathbf{W}^{-H}$ H $\mathbf{W}^{-1}$, $\psi(\mathbf{y}) = [\psi(\ _1), \ldots, \psi(\ _n)]^T$

:

$$\psi(\ _i) = -\frac{d \log\ (\ _i)}{d\ _i^*} = -\frac{\frac{\partial p(y_i)}{\partial y_i} + \frac{\partial p(y_i)}{\partial y_i}}{(\ _i)} = \frac{\partial \log\ (\ _i)}{\partial\ _i} + \frac{\partial \log\ (\ _i)}{\partial\ _i}, \quad (8)$$

$\psi(\cdot)$ k *complex score function*. H ,

$(\ _i) \equiv (\ _i, \ _i)$ k

(8) $_i$

$_i$.

(6), k (2)

- 7, 13 :

$$\Delta \mathbf{W} = \eta\big(\mathbf{I} - \psi(\mathbf{y})\mathbf{y}^H\big)\mathbf{W}. \quad (9)$$

Liouville's theorem, k *boundedness* *analyticity*

; $\psi(\cdot)$ (., 7 - 9 , 13 , 14); ,

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- . ,

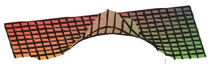
$e_i = \ _i - \ _i$, $H(\ _i \mid \ _i)$

$$H(\ _i \mid \ _i) = H(\ _i - \ _i) \equiv H(e_i) \quad (10)$$

, $H(\ _i) \equiv H(\ _i, \ _i)$ -

$$H(\ _i) = H(\ _i \mid \ _i) + H(\ _i) \equiv H(-e_i) + H(\ _i) = H(\ _i \mid \ _i) + H(\ _i) \equiv H(e_i) + H(\ _i) \quad (11)$$

e_i - ; $_i$ $_i$



ij ik . , i, $ij \neq 0$. $Q(\ i) > 0$ **R** , **R** ; **R** **P** **D**. □

Corollary 1: Q

$$Q(\ _1 + \ _2) = Q(\ _1) + Q(\ _2) \tag{19}$$

$_1$ $_2$; (17) ; $Q(\ _j) > 0$ $_j$.

Proof: (19) . Q , (19)

$$\begin{aligned} Q^2(\ _1 + \ _2) &= Q^2(\ _1) + Q^2(\ _2) + 2Q(\ _1)Q(\ _2) \\ &\geq Q^2(\ _1) + Q^2(\ _2), \end{aligned}$$

$Q(\ _1) = Q(\ _2) = 0$. 2. □

2 1 *sufficient* , k . H , k (., 4 , 15) A A . , - (> 2) *not* $\text{cum}_k(\ _1 + \ _2) = \text{cum}_k(\ _1) + \text{cum}_k(\ _2)$ $_1$ $_2$. , $_1 = |\ _1|e^{j\theta_1}$, $_2 = |\ _2|e^{j\theta_2}$ $(\theta_1, \theta_2 \in \mathbb{R})$,

$$\begin{aligned} _1 + \ _2 &= |\ _1|e^{j\theta_1} + |\ _2|e^{j\theta_2} \\ &= \sqrt{|\ _1|^2 + |\ _2|^2 + 2|\ _1||\ _2|\cos(\theta_1 - \theta_2)}e^{j\theta}, \end{aligned}$$

$\theta = \arctan \frac{_1 \ \mathrm{n}\,\theta_1 + \ _2 \ \mathrm{n}\,\theta_2}{_1 \ \theta_1 + \ _2 \ \theta_2}$. , $\theta_1 - \theta_2$, $||\ _1| - |\ _2|| \leq |\ _1 + \ _2| \leq |\ _1| + |\ _2|$.

Theorem 3: Q (2-) ; $= |\ |e^{j\theta}$ $(\theta \in \mathbb{R})$, $\overline{Q}(\ ) = Q(|\ |) = \sqrt{Q^2(|\ |\cos\theta) + Q^2(|\ |\sin\theta)}$ (2-) .

Proof: $_1 = |\ _1|e^{j\theta_1}$ $_2 = |\ _2|e^{j\theta_2}$; $= |\ |e^{j\theta} = \ _1 + \ _2$, $=$ $_1 + \ _2$ $= \ _1 + \ _2$; Q ,

$$\begin{aligned} Q^2(|\ |\cos\theta) &\geq Q^2(|\ _1|\cos\theta_1) + Q^2(|\ _2|\cos\theta_2) \\ Q^2(|\ |\sin\theta) &\geq Q^2(|\ _1|\sin\theta_1) + Q^2(|\ _2|\sin\theta_2). \end{aligned}$$

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$$\begin{aligned} \overline{Q}^2(\ ) &= Q^2(|\ |) = Q^2(|\ |\cos\theta) + Q^2(|\ |\sin\theta) \\ &\geq Q^2(|\ _1|) + Q^2(|\ _2|) = \overline{Q}^2(\ _1) + \overline{Q}^2(\ _2), \end{aligned} \tag{20}$$

5 . , 2- . □

H , , , - , 3 .

Lemma 1: $Q(\ )$ $(\ \in \mathbb{C})$ - ; $\tilde{\mathbf{z}} = [\ ,\ ]$, Q , $\tilde{\mathbf{z}} \in \mathbb{R}^2$; , ( ) : $Q(\boldsymbol{\alpha} + \tilde{\mathbf{z}}) = Q(\tilde{\mathbf{z}})$ $(\forall \boldsymbol{\alpha} \in \mathbb{R}^2)$; ( ) : $Q(\alpha\tilde{\mathbf{z}}) = |\alpha|Q(\tilde{\mathbf{z}})$ $(\forall \alpha \in \mathbb{R})$.

Proof: . , $\alpha = |\alpha|e^{j0}$ (. .,). , . □

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Theorem 4: Q - , ≥ 2, $-\sum_{i}^{n}\ _1 Q^k(\ _i)$ $-\sum_{i}^{n}\ _1 Q^{2k}(\ _i)$, - ; > 2 , $\{\ _j\}$, $Q(\ _j) = 0$.

Proof: 15 k $= 2$ $\mathbf{R} = \mathbf{WA}$, , - . Q, $Q^2(\ _i) \leq \sum_{j}^{n}\ _1 |\ _{ij}|^2 Q^2(\ _j) \leq \sum_{j}^{n}\ _1 Q^2(\ _j)$ $|\ _{ij}| \leq \sum_{k}^{n}\ _1 |\ _{ik}|^2 = 1$; > 2, $Q^k(\ _i) \leq \sum_{j}^{n}\ _1 |\ _{ij}|^k Q^k(\ _j) \leq \sum_{j}^{n}\ _1 Q^k(\ _j)$. $-\sum_{i}^{n}\ _1 Q^k(\ _i) \geq -\sum_{j}^{n}\ _1 Q^k(\ _j)$, $-\sum_{i}^{n}\ _1 Q^k(\ _i)$. k . □

Remark: : $\mathbb{E}[\mathbf{x}\mathbf{x}^H] = \mathbf{A}\mathbb{E}[\mathbf{s}\mathbf{s}^H]\mathbf{A}^H = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^H$, $\mathbf{\Sigma}$, $\mathbf{U}$, - , . $\mathbf{z} = \mathbf{D}^{-1/2}\mathbf{U}^H\mathbf{x} = \mathbf{D}^{-1/2}\mathbf{U}^H\mathbf{A}\mathbf{s} \equiv \tilde{\mathbf{A}}\mathbf{s}$, $\tilde{\mathbf{A}} = \mathbf{D}^{-1/2}\mathbf{U}^H\mathbf{A}$, $\mathbb{E}[\mathbf{z}\mathbf{z}^H] = \tilde{\mathbf{A}}\mathbb{E}[\mathbf{s}\mathbf{s}^H]\tilde{\mathbf{A}}^H = \mathbf{I}$; , $\tilde{\mathbf{A}}$, $\mathbb{E}[\mathbf{s}\mathbf{s}^H] = \mathbf{I}$.

B. Examples of Contrast Functions

1) Range Function: 15 21 , - A, . A , . ., , , . , ,

$R( )$,

:

$$R( ) = d, \quad \{ ( ) \geq 0 \; \big| \; | - {}_0| \leq d\} \tag{21}$$

$_0$, $d \in \mathbb{R}^+$.

Theorem 5: $_1$ $_2$,

$$R({}_1 + {}_2) = R({}_1) + R({}_2). \tag{22}$$

Proof: $| {}_1 + {}_2| \leq | {}_1| + | {}_2|$. □

Corollary 2: $= \alpha {}_1 + \beta {}_2$, $_1$ $_2$, $\alpha, \beta \in \mathbb{C}$,

$$R( ) = |\alpha| \cdot R({}_1) + |\beta| \cdot R({}_2). \tag{23}$$

5 $\alpha {}_1$ ($\beta {}_2$) $|\alpha|$ ($|\beta|$) , $_1$ ($_2$), α (β).

$Q({}_i) = R({}_i)$, (17)

$$J(\mathbf{W}) = \sum_{i\ 1}^{n} \log\Big(\sum_{j\ 1}^{n} | {}_{ij}|R({}_j)\Big) - \log| \ (\mathbf{W})|. \tag{24}$$

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Definition 6: , $\{ : | | \leq +\infty\}$; $\int_0^d (| |)d| | \approx 1 \quad (0 < d < +\infty)$. , d .

2) Shannon Entropy Function: A -k - , $H({}_i)$.

$$\overline{Q}({}_i) = Q(| {}_i|) = \exp(H(| {}_i|)), \tag{25}$$

entropy power inequality, k (25) . 3 . (25) (17) (6) . $\{ {}_i\}$, $\{| {}_i|\}$ / - / , k 16 , 17 .

3) Rényi Entropy Function: - , - $(0 < \ \in \mathbb{R})$:

$$H_k({}_i) = \frac{1}{1 - } \log\Big(\int ({}_i)^k d {}_i\Big). \tag{26}$$

$\to 1$ k , . $= 2$, 2 *extension entropy*

$$H_2({}_i) = -\log\Big(\int ({}_i)^2 d {}_i\Big).$$

Jensen inequality, $H_2({}_i) \leq H({}_i)$. , - $H_k({}_i) \geq H_r({}_i)$ $>$.

Lemma 2: $\{ : ( ) = g(| |)\}$, , $\exp(H_2( ))$, .

Proof: , $H_2( ) = H_2(+ \alpha)$ α. ,

$$H_2(\alpha) = -\log\Big(\int g(|\alpha \ |)^2 d|\alpha \ |\Big)$$
$$= -\log\Big(\int g(| \ |)^2 d| \ |\Big) + \log|\alpha|.$$

$\exp(H_2(\alpha \ )) = |\alpha| \exp(H_2(\ ))$. , $\exp(H_2(\ ))$. □

, $\overline{Q}({}_i) = Q(| {}_i|) = \exp(H_2(| {}_i|)) = \frac{1}{\int p(|y_i|)^2 dy_i}$, 2- . , 26,27 ; $\overline{Q}$. $\{| {}_i|\}$, k 18 .

4) Fisher Information Function:

$$\mathbf{G} = \mathbb{E}[\psi(\ )^2] = \mathbb{E}\Big[\Big(\frac{d \log \ (\ )}{d}\Big)^2\Big], \tag{27}$$

$\psi(\ ) = \frac{d \quad p(\ )}{d} = \frac{p'(\ )}{p(\ )}$. $Q(\ ) = \mathbf{G}^{-1/2}$, Q 15 . ,

$$\overline{Q}(\ ) = \mathbb{E}\Big[\Big(\frac{\ '(| \ |)}{(| \ |)}\Big)^2\Big]^{-1/2}.$$

- , - $\psi_k(\ ) =$

$$Q(\ ) = (\Sigma\ )^{1/2} \equiv \left(\mathrm{cov}[\tilde{\mathbf{z}}]\right)^{1/2},$$

$$\begin{aligned} Q(\omega\ ) &= \left(\mathrm{cov}[\tilde{\mathbf{z}}']\right)^{1/2} = \left(\mathrm{cov}[\Omega\tilde{\mathbf{z}}]\right)^{1/2} \\ &= \left(\Omega\mathrm{cov}[\tilde{\mathbf{z}}]\right)^{1/2} = (\Omega)^{1/2} \left(\mathrm{cov}[\tilde{\mathbf{z}}]\right)^{1/2} \\ &= (\omega^2 + \omega^2\ )^{1/2} (\Sigma\ )^{1/2} = |\omega| (\Sigma\ )^{1/2}. \end{aligned}$$

$Q(\ )$, . ,
$Q(\ )$; , $Q(\ )$
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- A A, - ( ) - (k) 22 - 24 , A.
- A - , 25 (W). × $\mathcal{U}($), [2]. .
- A A , (. .,) , - . , .

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- A 7, 13 , 14 , - 5, 12 . A 6 - 14 . H , A , - . , k . , , k . ; .

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- (/ , ,), - - - -

$$( ) \approx \mathcal{N}( )\left(1 + \frac{\kappa^3}{ }\mathcal{H}_3( ) + \frac{\kappa^4}{24}\mathcal{H}_4( )\right)$$