

Contrast Functions for Non-circular and Circular Sources Separation in Complex-Valued ICA

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Abstract—In this paper, the complex-valued ICA problem is studied in the context of blind complex-source separation. We formulate the complex ICA problem in a general setting, and define the superadditive functional that may be used for constructing a contrast function for circular complex sources separation. We propose several contrast functions and study their properties. Finally, we also discuss relevant issues and present the convex analysis of a specific contrast function.

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$$\begin{aligned}
& \bullet \quad \text{Linearity of expectation:} \\
& \mathbb{E}[\mathbf{X}] = \mathbb{E}[\mathbf{X}_1 + \mathbf{X}_2] = \mathbb{E}[\mathbf{X}_1] + \mathbb{E}[\mathbf{X}_2]. \\
& \bullet \quad \text{Variance of a sum:} \\
& \mathbb{E}[\|\mathbf{X}\|^2] = \mathbb{E}[\|\mathbf{X}_1\|^2] + \mathbb{E}[\|\mathbf{X}_2\|^2] + 2 \mathbb{E}[\mathbf{X}_1^T \mathbf{X}_2]. \\
& \bullet \quad \text{Variance of a linear combination:} \\
& \text{var}[\mathbf{X}] = \mathbb{E}[\|\mathbf{X} - \mathbb{E}[\mathbf{X}]\|^2] = \mathbb{E}[\|\mathbf{X}\|^2] - \|\mathbb{E}[\mathbf{X}]\|^2. \\
& \bullet \quad \text{Covariance matrix:} \\
& C_{ij} = \mathbb{E}[(\mathbf{x}_i - \mathbb{E}[\mathbf{x}_i])(\mathbf{x}_j^* - \mathbb{E}[\mathbf{x}_j^*])] = \mathbb{E}[\mathbf{x}_i \mathbf{x}_j^*] - \mathbb{E}[\mathbf{x}_i] \mathbb{E}[\mathbf{x}_j^*]. \\
& \quad \text{uncorrelated} \quad C_{ij} = 0. \\
& \bullet \quad \text{Covariance matrix of a vector:} \\
& \mathbf{z} = [z_1, \dots, z_n]^T, \quad \mathbf{z}^H = [z_1^*, \dots, z_n^*]^T \equiv (\mathbf{z}^*)^T \\
& \quad \text{Cov}[\mathbf{z}] \equiv \mathbb{E}[(\mathbf{z} - \mathbb{E}[\mathbf{z}])(\mathbf{z} - \mathbb{E}[\mathbf{z}])^H]; \quad \text{pseudo-covariance matrix} \\
& \quad \text{pcov}[\mathbf{z}] \equiv \mathbb{E}[(\mathbf{z} - \mathbb{E}[\mathbf{z}])(\mathbf{z} - \mathbb{E}[\mathbf{z}])^T]. \\
& \text{Definition 1: A random vector } \mathbf{z} \text{ is } \alpha\text{-circular if} \\
& \quad \mathbb{E}[\mathbf{z} e^{j\alpha}] = e^{j\alpha} \mathbb{E}[\mathbf{z}], \quad \text{for all } \alpha. \\
& \text{order circular} \quad \mathbf{z} \text{ is second-order circular if} \\
& \quad \mathbb{E}[\mathbf{z}] = \mathbf{0}, \quad \mathbb{E}[\|\mathbf{z}\|^2] = 0 \\
& \quad \mathbb{E}[\mathbf{z} \mathbf{z}^H] \text{ is symmetric; } \mathbb{E}[\mathbf{z} \mathbf{z}^T] = \mathbf{0}. \\
& \quad \text{strongly uncorrelated} \quad \mathbb{E}[\mathbf{z} \mathbf{z}^H] = \mathbf{0}. \\
& \quad \text{zero-mean} \quad \mathbb{E}[\mathbf{z}] = \mathbf{0}.
\end{aligned}$$

$$\text{skewness}(\cdot) = \mathbb{E}[|\cdot|^3]/(\mathbb{E}[|\cdot|^2])^{3/2}, \quad (1)$$

$$\text{kurtosis}(\cdot) = \mathbb{E}[|\cdot|^4] - 2(\mathbb{E}[|\cdot|^2])^2 - |\mathbb{E}[\cdot^2]|^2. (2)$$

Definition 2: $\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{pmatrix}$ is called a \mathbf{A} -bidiagonal matrix if $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ are $n_1 \times n_1, n_1 \times n_2, n_2 \times n_1, n_2 \times n_2$ matrices, respectively, and $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ are upper triangular matrices.

$${}^1_1({}^2_1|, {}^2_2|) = ({}^1_1|)({}^2_2|),$$

$$J(\mathbf{z}) = \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N \sum_{m=1}^M \sum_{n=1}^M \sum_{p=1}^M \sum_{q=1}^M \sum_{r=1}^M \sum_{s=1}^M \sum_{t=1}^M \sum_{u=1}^M \sum_{v=1}^M \sum_{w=1}^M \sum_{x=1}^M \sum_{y=1}^M \sum_{z=1}^M \sum_{\mathbf{z} \in \mathbb{C}^N} J(\mathbf{z})$$

$$\nabla J \equiv \frac{\partial J(\mathbf{z})}{\partial \mathbf{z}^*} = \frac{1}{2} \left(\frac{\partial J(\mathbf{z})}{\partial \mathbf{z}} + \frac{\partial J(\mathbf{z})}{\partial \bar{\mathbf{z}}} \right) = 0.$$

$$, \frac{\partial J(\mathbf{z})}{\partial \mathbf{x}} = \frac{\partial J(\mathbf{z})}{\partial \mathbf{z}} = 0.$$

Definition 3: A \mathbf{z} -invariant function $J(\mathbf{z})$ ($\mathbf{z} \in \mathbb{C}^N$) is called a \mathbf{z} -invariant function if it satisfies

$$J(\lambda \mathbf{z}_1 + (1 - \lambda) \mathbf{z}_2) \leq \lambda J(\mathbf{z}_1) + (1 - \lambda) J(\mathbf{z}_2)$$

$$\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{C}^N \quad 0 \leq \lambda \leq 1.$$

$$\mathbf{H} = \frac{\partial^2 J(\mathbf{z})}{\partial \mathbf{z} \partial \mathbf{z}^H},$$

$$J(\mathbf{z})$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\mathbf{s} \in \mathbb{C}^n$$

$$\mathbf{A} \in \mathbb{C}^{n \times n} \quad (\text{..,})$$

A:

- $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
- $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$
- $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$

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the β phase of the polymer. The β phase is the more ordered phase and is characterized by a higher density and a higher melting point than the α phase. The β phase is also the more stable phase and is the one that is most commonly observed in nature. The α phase is the less ordered phase and is characterized by a lower density and a lower melting point than the β phase. The α phase is also the less stable phase and is the one that is most commonly observed in nature.

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$$(6), \quad \quad \quad (\quad \quad 2)$$

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$$\mathbb{E}[\mathbf{x}] = 0 \quad \mathbb{E}[\mathbf{x}\mathbf{x}^H] = \mathbf{I},$$

$$\Delta \mathbf{W} = \eta (\mathbf{I} - \psi(\mathbf{y})\mathbf{y}^H) \mathbf{W}. \tag{9}$$

strong-uncorrelating transform 10, 11 .

Liouville's theorem,
boundedness analyticity

formation, mutual in- ; $\psi(\cdot)$ (. . . , 7 -
9 , 13 , 14);

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$$\begin{aligned} I(\textcolor{teal}{i}_1, \dots, \textcolor{teal}{i}_n) &= \mathbb{E}_{p(\mathbf{y})} \Big[\log \frac{(\mathbf{y})}{\prod_{i=1}^n (\textcolor{teal}{i}_i)} \Big] \\ &= \int (\mathbf{y}) \log \frac{(\mathbf{y})}{\prod_{i=1}^n (\textcolor{teal}{i}_i)} d\mathbf{y} \end{aligned} \tag{4}$$

$$\begin{aligned} e_i &= \textcolor{teal}{i}_i - \textcolor{teal}{i}_i, \\ H(\textcolor{teal}{i}_i \mid \textcolor{teal}{i}_i) &= H(\textcolor{teal}{i}_i - \textcolor{teal}{i}_i) \equiv H(e_i) \end{aligned} \tag{10}$$

$\{ \textcolor{teal}{i}_1, \dots, \textcolor{teal}{i}_n \}$
 $\{ \textcolor{teal}{i}_1, \dots, \textcolor{teal}{i}_n \}$

$$H(\textcolor{teal}{i}_i) \equiv H(\textcolor{teal}{i}_i, \textcolor{teal}{i}_i)$$

$$I(\textcolor{teal}{i}_1, \dots, \textcolor{teal}{i}_n) = -\frac{1}{2} \log \Big(\frac{(\mathbf{C}_{\mathbf{y}})}{\prod_{i=1}^n C_{ii}} \Big) = -\frac{1}{2} \sum_{i=1}^n \log(\lambda_i) \tag{5}$$

$$\begin{aligned} H(\textcolor{teal}{i}_i) &= H(\textcolor{teal}{i}_i \mid \textcolor{teal}{i}_i) + H(\textcolor{teal}{i}_i) \equiv H(-e_i) + H(\textcolor{teal}{i}_i) \\ &= H(\textcolor{teal}{i}_i \mid \textcolor{teal}{i}_i) + H(\textcolor{teal}{i}_i) \equiv H(e_i) + H(\textcolor{teal}{i}_i) \end{aligned} \tag{11}$$

$$\begin{aligned} C_{ij} &= \mathbb{E}[(\textcolor{teal}{i}_i - \mathbb{E}[\textcolor{teal}{i}_i])(\textcolor{teal}{j}_j^* - \mathbb{E}[\textcolor{teal}{j}_j^*])], \quad \lambda_i \\ \mathbf{C}_{\mathbf{y}} &= \text{cov}[\mathbf{y}], \quad \mathbf{\Lambda} = \begin{Bmatrix} C_{11}, C_{22}, \dots, C_{nn} \end{Bmatrix}. \end{aligned}$$

$$e_i \quad ; \quad \textcolor{teal}{i}_i \quad \textcolor{teal}{i}_i$$

$$I(\textcolor{teal}{i}_1, \dots, \textcolor{teal}{i}_n) = \sum_{i=1}^n H(\textcolor{teal}{i}_i) - H(\mathbf{y}),$$

$$\begin{aligned} H(\mathbf{y}) &= H(\mathbf{x}) + \log |\textcolor{teal}{i}_1, \dots, \textcolor{teal}{i}_n|, \\ H(\mathbf{y}) &= H(\mathbf{x}) + \log |\textcolor{teal}{i}_1, \dots, \textcolor{teal}{i}_n|, \end{aligned} \tag{4}$$

$$J(\mathbf{W}) = \sum_{i=1}^n H(\textcolor{teal}{i}_i) - \log |\textcolor{teal}{i}_1, \dots, \textcolor{teal}{i}_n| - H(\mathbf{x}), \tag{6}$$

$$\begin{aligned} H(\mathbf{x}) &= H(\mathbf{x}) + \log |\textcolor{teal}{i}_1, \dots, \textcolor{teal}{i}_n|, \\ \mathbf{W}^* &= (\dots) \mathbf{W}. \end{aligned} \tag{6} \dots$$

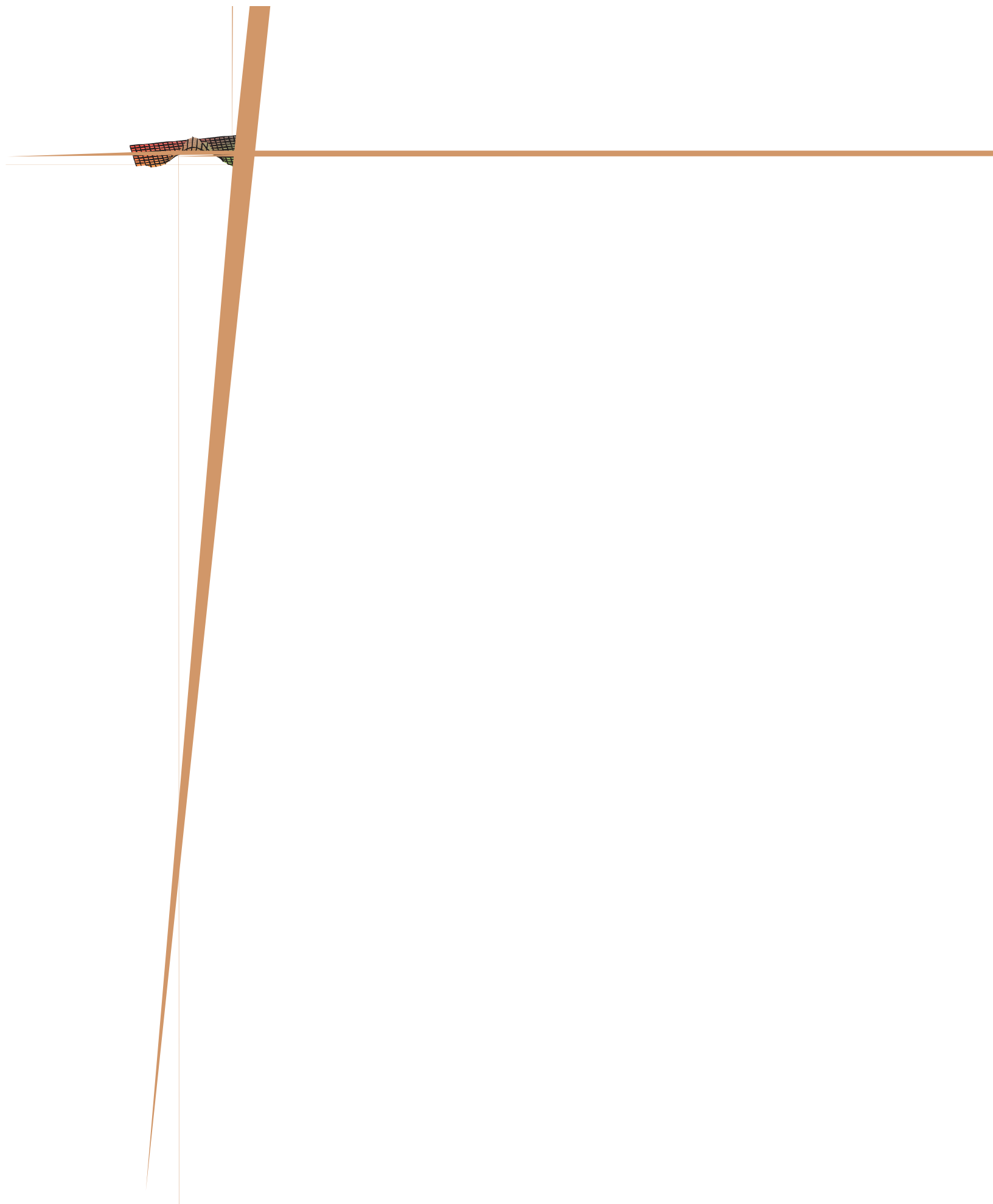
$$\nabla_{\mathbf{W}^*} J(\mathbf{W}) = \left(\mathbb{E}_{\mathbf{y}} [\psi(\mathbf{y})\mathbf{y}^H] - \mathbf{I} \right) \mathbf{W}^{-H}, \tag{7}$$

$$\begin{aligned} \mathbf{W}^{-H} &= \mathbf{W}^{-1}, \\ \psi(\mathbf{y}) &= [\psi(\textcolor{teal}{i}_1), \dots, \psi(\textcolor{teal}{i}_n)]^T \end{aligned}$$

$$\begin{aligned} \psi(\textcolor{teal}{i}_i) &= -\frac{d \log (\textcolor{teal}{i}_i)}{d \textcolor{teal}{i}_i^*} = -\frac{\frac{\partial p(y_i)}{\partial y_i} + \frac{\partial p(y_i)}{\partial y_i}}{(\textcolor{teal}{i}_i)} \\ &= \frac{\partial \log (\textcolor{teal}{i}_i)}{\partial \textcolor{teal}{i}_i} + \frac{\partial \log (\textcolor{teal}{i}_i)}{\partial \textcolor{teal}{i}_i}, \end{aligned} \tag{8}$$

$$\begin{aligned} \psi(\cdot) &= \text{complex score function.} \\ (\textcolor{teal}{i}_i) &\equiv (\textcolor{teal}{i}_i, \textcolor{teal}{i}_i) \end{aligned} \tag{8}$$

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$$Q(\mathbf{1} + \mathbf{2}) = Q(\mathbf{1}) + Q(\mathbf{2}) \quad (19)$$

$$Q(j) > 0$$

$$\begin{aligned} Q^2(\mathbf{v}_1 + \mathbf{v}_2) &= Q^2(\mathbf{v}_1) + Q^2(\mathbf{v}_2) + 2Q(\mathbf{v}_1)Q(\mathbf{v}_2) \\ &\geq Q^2(\mathbf{v}_1) + Q^2(\mathbf{v}_2), \end{aligned}$$

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$\wedge (\quad) > 2)$

not $\text{cum}_k(\quad_1 + \quad_2) = \text{cum}_k(\quad_1) + \text{cum}_k(\quad_2)$

$\quad_1 \quad_2$

$$\begin{aligned} |1\rangle + |2\rangle &= |1\rangle e^{j\theta_1} + |2\rangle e^{j\theta_2} \\ &= \sqrt{|1|^2 + |2|^2 + 2|1||2|\cos(\theta_1 - \theta_2)} e^{j\theta} \end{aligned}$$

$$||\mathbf{1}| - |\mathbf{2}|| \leq$$

Theorem 3: Q is a 2-regular graph; $\mathbf{Q} = |e^{j\theta}|$ ($\theta \in \mathbb{R}$),

Proof: $\mathbf{1} = \mathbf{1} e^{j\theta_1}$
 $\mathbf{2} = \mathbf{2} e^{j\theta_2}$
 $\mathbf{1} + \mathbf{2} = \mathbf{1} e^{j\theta} = \mathbf{1} + \mathbf{2}$,
 \mathbf{Q}

$$\begin{aligned}\overline{Q}^2(\mathbf{r}) &= Q^2(|\mathbf{r}|) = Q^2(|\mathbf{r}|\cos\theta) + Q^2(|\mathbf{r}|\sin\theta) \\ &\geq Q^2(|\mathbf{r}_1|) + Q^2(|\mathbf{r}_2|) = \overline{Q}^2(\mathbf{r}_1) + \overline{Q}^2(\mathbf{r}_2),\end{aligned}\quad (20)$$

$$\begin{aligned} \tilde{\mathbf{z}} &\in \mathbb{R}^2; & (1) \\ Q(\boldsymbol{\alpha} + \tilde{\mathbf{z}}) &= Q(\tilde{\mathbf{z}}) \ (\forall \boldsymbol{\alpha} \in \mathbb{R}^2); & (2) \\ Q(\alpha \tilde{\mathbf{z}}) &= |\alpha| Q(\tilde{\mathbf{z}}) \ (\forall \alpha \in \mathbb{R}). \end{aligned}$$

$|\alpha|e^{j0}$ (.,.,.).? $\alpha =$. \square

Theorem 4: $\mathcal{Q}(\mathbf{A}) \geq 2$, $\mathcal{Q}(\mathbf{A}) = 2$ if and only if \mathbf{A} is a 2×2 matrix with $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ or $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

[illegible]
$$\begin{aligned} \mathbf{A}\mathbb{E}[\mathbf{ss}^H]\mathbf{A}^H &= \mathbf{U}\Sigma\mathbf{U}^H, & \Sigma &= \mathbf{D}^{-1/2}\mathbf{U}^H\mathbf{U}\mathbf{D}^{-1/2} : \mathbb{E}[\mathbf{xx}^H] = \mathbf{I}, \\ \mathbf{D}^{-1/2}\mathbf{U}^H\mathbf{A}\mathbf{s} &\equiv \tilde{\mathbf{A}}\mathbf{s}, & \tilde{\mathbf{A}} &= \mathbf{D}^{-1/2}\mathbf{U}^H\mathbf{A} \\ \mathbb{E}[\mathbf{zz}^H] &= \tilde{\mathbf{A}}\mathbb{E}[\mathbf{ss}^H]\tilde{\mathbf{A}}^H = \mathbf{I}, & \mathbb{E}[\mathbf{ss}^H] &= \mathbf{I}. \end{aligned}$$

1) Range Function:

$$R(\cdot) = d, \quad \{(\cdot) \geq 0 \mid |\cdot - 0| \leq d\} \quad (21)$$

Theorem 5: $d \in \mathbb{R}^+$

$$R(\cdot_1 + \cdot_2) = R(\cdot_1) + R(\cdot_2). \quad (22)$$

Proof: $|\cdot_1 + \cdot_2| \leq | \cdot_1 | + | \cdot_2 |$ \square

Corollary 2: $\alpha \cdot_1 + \beta \cdot_2$

$$R(\cdot) = |\alpha| \cdot R(\cdot_1) + |\beta| \cdot R(\cdot_2). \quad (23)$$

$$|\beta| \cdot \alpha \cdot_1 (\cdot_2) = |\alpha| \cdot (\cdot_2), \quad (17)$$

$$Q(\cdot_i) = R(\cdot_i), \quad (17)$$

$$J(\mathbf{W}) = \sum_{i=1}^n \log \left(\sum_{j=1}^n |_{ij}| R(\cdot_j) \right) - \log |(\mathbf{W})|. \quad (24)$$

Definition 6: $\{ \cdot : | \cdot | \leq +\infty \}$
 $\int_0^d (| \cdot |) d| \approx 1 \quad (0 < d < +\infty).$

2) **Shannon Entropy Function:** A $H(\cdot_i).$

$$\overline{Q}(\cdot_i) = Q(| \cdot_i |) = \exp(H(| \cdot_i |)), \quad (25)$$

$$\text{entropy power inequality}, \quad (25)$$

$$(25) \quad (17) \quad (6)$$

$$\{ | \cdot_i | \}, \quad \{ | \cdot_i | \}$$

$$16, 17.$$

3) **Rényi Entropy Function:** $(0 < \cdot \in \mathbb{R})$:

$$H_k(\cdot_i) = \frac{1}{1 - \cdot} \log \left(\int (\cdot_i)^k d \cdot_i \right). \quad (26)$$

$$\rightarrow 1, \quad = 2,$$

2 *extension entropy*

$$H_2(\cdot_i) = -\log \left(\int (\cdot_i)^2 d \cdot_i \right).$$

$$\text{Jensen inequality}, \quad H_2(\cdot_i) \leq H(\cdot_i).$$

$$H_k(\cdot_i) \geq H_r(\cdot_i) >$$

Lemma 2: $\{ \cdot : (\cdot) = g(| \cdot |) \}$, $\exp(H_2(\cdot))$,

$$\text{Proof:} \quad H_2(\cdot) = H_2(\cdot + \alpha)$$

$$H_2(\alpha \cdot) = -\log \left(\int g(| \alpha \cdot |)^2 d | \alpha \cdot | \right) = -\log \left(\int g(| \cdot |)^2 d | \cdot | \right) + \log |\alpha|.$$

$$\exp(H_2(\alpha \cdot)) = |\alpha| \exp(H_2(\cdot)). \quad \square$$

$$\exp(H_2(| \cdot_i |)) = \frac{1}{\int p(|y_i|)^2 dy_i}, \quad \overline{Q}(\cdot_i) = Q(| \cdot_i |) =$$

$$26, 27; \quad \overline{Q} \quad \{ | \cdot_i | \},$$

4) **Fisher Information Function:**

$$\mathbf{G} = \mathbb{E}[\psi(\cdot)^2] = \mathbb{E} \left[\left(\frac{d \log(\cdot)}{d} \right)^2 \right], \quad (27)$$

$$\psi(\cdot) = \frac{d \cdot p(\cdot)}{d} = \frac{p'(\cdot)}{p(\cdot)}, \quad Q(\cdot) = \mathbf{G}^{-1/2}, \quad Q$$

$$15. \quad \overline{Q}(\cdot) = \mathbb{E} \left[\left(\frac{p'(| \cdot |)}{p(| \cdot |)} \right)^2 \right]^{-1/2}$$

$$\psi_k(\cdot) =$$

$$Q(\mathbf{\Sigma}) = \left(\mathbf{\Sigma}\right)^{1/2} \equiv \left(\text{cov}[\tilde{\mathbf{z}}]\right)^{1/2},$$

$$\begin{aligned} Q(\omega) &= \left(\text{cov}[\tilde{\mathbf{z}}']\right)^{1/2} = \left(\text{cov}[\Omega\tilde{\mathbf{z}}]\right)^{1/2} \\ &= \left(\Omega\text{cov}[\tilde{\mathbf{z}}]\right)^{1/2} = (\Omega)^{1/2} \left(\text{cov}[\tilde{\mathbf{z}}]\right)^{1/2} \\ &= (\omega^2 + \omega^2)^{1/2} (\mathbf{\Sigma})^{1/2} = |\omega| (\mathbf{\Sigma})^{1/2}. \end{aligned}$$

$$Q(\mathbf{\Sigma}) = \left(\mathbf{\Sigma}\right)^{1/2} = \left(\text{cov}[\tilde{\mathbf{z}}]\right)^{1/2}; \quad Q(\omega) = \left(\omega^2 + \omega^2\right)^{1/2} \left(\mathbf{\Sigma}\right)^{1/2} = |\omega| \left(\mathbf{\Sigma}\right)^{1/2}.$$

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$$\begin{aligned} & \left(\frac{1}{\kappa} \right) \approx \mathcal{N}(\kappa) \left(1 + \frac{\kappa^3}{2} \mathcal{H}_3(\kappa) + \frac{\kappa^4}{2} \mathcal{H}_4(\kappa) \right) \left(\frac{1}{\kappa} \right) \end{aligned}$$