Contrast Functions for Non-circular and Circular Sources Separation in Complex-Valued ICA

Abstract—**In this paper, the complex-valued ICA problem is studied in the context of blind complex-source separation. We formulate the complex ICA problem in a general setting, and define the superadditive functional that may be used for constructing a contrast function for circular complex sources separation. We propose several contrast functions and study their properties. Finally, we also discuss relevant issues and present the convex analysis of a specific contrast function.**

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$$\mathbb{E}[\ ] = \mathbb{E}[\ \ + \ \ ] = \mathbb{E}[\ \ ] + \ \mathbb{E}[\ \ ].$$

- - :

$$\mathbb{E}[\ ^2] = \mathbb{E}[\ ^2\ ] - \mathbb{E}[\ ^2\ ] + 2\ \mathbb{E}[\ \ \ ].$$

- - ( ):

$$\mathtt{var}[\ ] = \mathbb{E}[|\ \ - \mathbb{E}[\ ]|^2] = \mathbb{E}[|\ |^2] - \big|\mathbb{E}[\ ]\big|^2.$$

$C_{ij} = \mathbb{E}\big[(\ _i - \mathbb{E}[\ _i])(\ _j^* - \mathbb{E}[\ _j^*])\big] = \mathbb{E}[\ _i\ _j^*] - \mathbb{E}[\ _i]\mathbb{E}[\ _j^*]$. $_i$ $_j$ ($\neq i$) *uncorrelated* $C_{ij} = 0$.

$\mathbf{z} = [\ _1, \ldots, \ _n]$ - , $\mathbf{z}^H = [\ _1^*, \ldots, \ _n^*]^T \equiv (\mathbf{z}^*)^T$ (. .,), $\mathbb{E}[\mathbf{z}]$, $\mathtt{cov}[\mathbf{z}] \equiv \mathbb{E}\big[(\mathbf{z} - \mathbb{E}[\mathbf{z}])(\mathbf{z} - \mathbb{E}[\mathbf{z}])^H\big]$; , *pseudo-covariance matrix* $\mathtt{pcov}[\mathbf{z}] \equiv \mathbb{E}\big[(\mathbf{z} - \mathbb{E}[\mathbf{z}])(\mathbf{z} - \mathbb{E}[\mathbf{z}])^T\big]$.

Definition 1: A - α, ( ) ($e^{j\alpha}$) (. ., ( )). - $\mathbf{z}$ *second-order circular* - (. .,); $\mathbb{E}[\ ] = 0$, - $\mathbb{E}[\ ^2] = 0$

. $\mathbb{E}[\mathbf{z}\mathbf{z}^H]$, $\mathbf{z}$; $\mathbf{z}$ *strongly uncorrelated* $\mathbb{E}[\mathbf{z}\mathbf{z}^H]$ $\mathbb{E}[\mathbf{z}\mathbf{z}^T]$. $\mathbf{z}$, $\mathbf{z}$ *symmetric*; $\mathbf{z}$, $\mathbf{z}$ - , $\mathbb{E}[\mathbf{z}\mathbf{z}^T] = \mathbf{0}$.

, *zero-mean* 1 :

$$\mathtt{skewness}(\ ) = \mathbb{E}[|\ |^3]/\big(\mathbb{E}[|\ |^2]\big)^{3/2}, \quad (1)$$

$$\mathtt{kurtosis}(\ ) = \mathbb{E}[|\ |\ ] - 2\big(\mathbb{E}[|\ |^2]\big)^2 - \big|\mathbb{E}[\ ^2]\big|^2. \quad (2)$$

Definition 2: $(\ _1,\ _2) = \ ^1(\ _1)\ ^2(\ _2)$; ^{1} ^{2} , $(|\ _1|, |\ _2|) = (|\ _1|)\ (|\ _2|)$.

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$J(\mathbf{z})$ - , - $\mathbf{z} \in \mathbb{C}^N$, :

$$\nabla J \equiv \frac{\partial J(\mathbf{z})}{\partial \mathbf{z}^*} = \frac{1}{2}\Big(\frac{\partial J(\mathbf{z})}{\partial \mathbf{z}} + \ \frac{\partial J(\mathbf{z})}{\partial \mathbf{z}}\Big) = 0.$$

, $\frac{\partial J(\mathbf{z})}{\partial \mathbf{x}} = \frac{\partial J(\mathbf{z})}{\partial \mathbf{z}} = 0$.

Definition 3: A - $J(\mathbf{z})$ ($\mathbf{z} \in \mathbb{C}^N$)

$$J\big(\lambda\mathbf{z}_1 + (1-\lambda)\mathbf{z}_2\big) \le \lambda J(\mathbf{z}_1) + (1-\lambda)J(\mathbf{z}_2)$$

$\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{C}^N$ $0 \le \lambda \le 1$. A , , $\mathbf{H} = \frac{\partial^2 J(\mathbf{z})}{\partial \mathbf{z}\partial \mathbf{z}^H}$, (. .,), $J(\mathbf{z})$.

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A : $\mathbf{x} = \mathbf{A}\mathbf{s}$, $\mathbf{s} \in \mathbb{C}^n$ - - , - , $\mathbf{x} \in \mathbb{C}^n$ - - , $\mathbf{A} \in \mathbb{C}^{n \times n}$ (. .,) - .

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$\mathbb{E}[\mathbf{x}] = 0$ $\mathbb{E}[\mathbf{x}\mathbf{x}^H] = \mathbf{I}$,

strong-uncorrelating transform 10, 11 .

mutual information,

:

$$I(_1,\ldots, _n) = \mathbb{E}_{p(\mathbf{y})}\Big[\log \frac{(\mathbf{y})}{\prod_{i\ 1}^n (_i)}\Big] = \int (\mathbf{y}) \log \frac{(\mathbf{y})}{\prod_{i\ 1}^n (_i)} d\mathbf{y} \quad (4)$$

(4)

$\{ _1,\ldots, _n\}$.

$\{ _1,\cdots, _n\}$,

(4)

$$I(_1,\ldots, _n) = -\frac{1}{2}\log\Big(\frac{(\mathbf{C_y})}{\prod_{i\ 1}^n C_{ii}}\Big) = -\frac{1}{2}\sum_{i\ 1}^n \log(\lambda_i) \quad (5)$$

$C_{ij} = \mathbb{E}[(_i - \mathbb{E}[_i])(_j^* - \mathbb{E}[_j^*])]$, λ_i

0 : $\mathbf{C_y u} = \lambda \mathbf{\Lambda u}$,

$\mathbf{C_y} = \mathrm{cov}[\mathbf{y}]$, $\mathbf{\Lambda} =$ $\{C_{11}, C_{22},\ldots, C_{nn}\}$.

$$I(_1,\ldots, _n) = \sum_{i\ 1}^n H(_i) - H(\mathbf{y}),$$

$H(\mathbf{y})$ -

$\{ _1,\ldots, _n\}$; $H(\mathbf{y}) = H(\mathbf{x}) + \log|\ (\mathbf{W})|$, (4)

$$J(\mathbf{W}) = \sum_{i\ 1}^n H(_i) - \log|\ (\mathbf{W})| - H(\mathbf{x}), \quad (6)$$

$H(\mathbf{x})$

(. . .) $\mathbf{W}$. (6) . . .

$\mathbf{W}^*$

$$\nabla_{\mathbf{W}^*} J(\mathbf{W}) = \Big(\mathbb{E}_{\mathbf{y}}[\psi(\mathbf{y})\mathbf{y}^H] - \mathbf{I}\Big)\mathbf{W}^{-H}, \quad (7)$$

$\mathbf{W}^{-H}$ $\mathbf{W}^{-1}$,

$\psi(\mathbf{y}) = [\psi(_1),\ldots,\psi(_n)]^T$

:

$$\psi(_i) = -\frac{d \log (_i)}{d _i^*} = -\frac{\frac{\partial p(y_i)}{\partial y_i} + \frac{\partial p(y_i)}{\partial y_i}}{(_i)} = \frac{\partial \log (_i)}{\partial _i} + \frac{\partial \log (_i)}{\partial _i}, \quad (8)$$

$\psi(\cdot)$ *complex score function.* ,

$(_i) \equiv (_i, _i)$

(8) $_i$

$_i$.

(6), (2)

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7, 13 :

$$\Delta\mathbf{W} = \eta\big(\mathbf{I} - \psi(\mathbf{y})\mathbf{y}^H\big)\mathbf{W}. \quad (9)$$

Liouville's theorem,

boundedness *analyticity*

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$\psi(\cdot)$ (. ., 7 - 9 , 13 , 14); ,

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$e_i = _i - _i$,

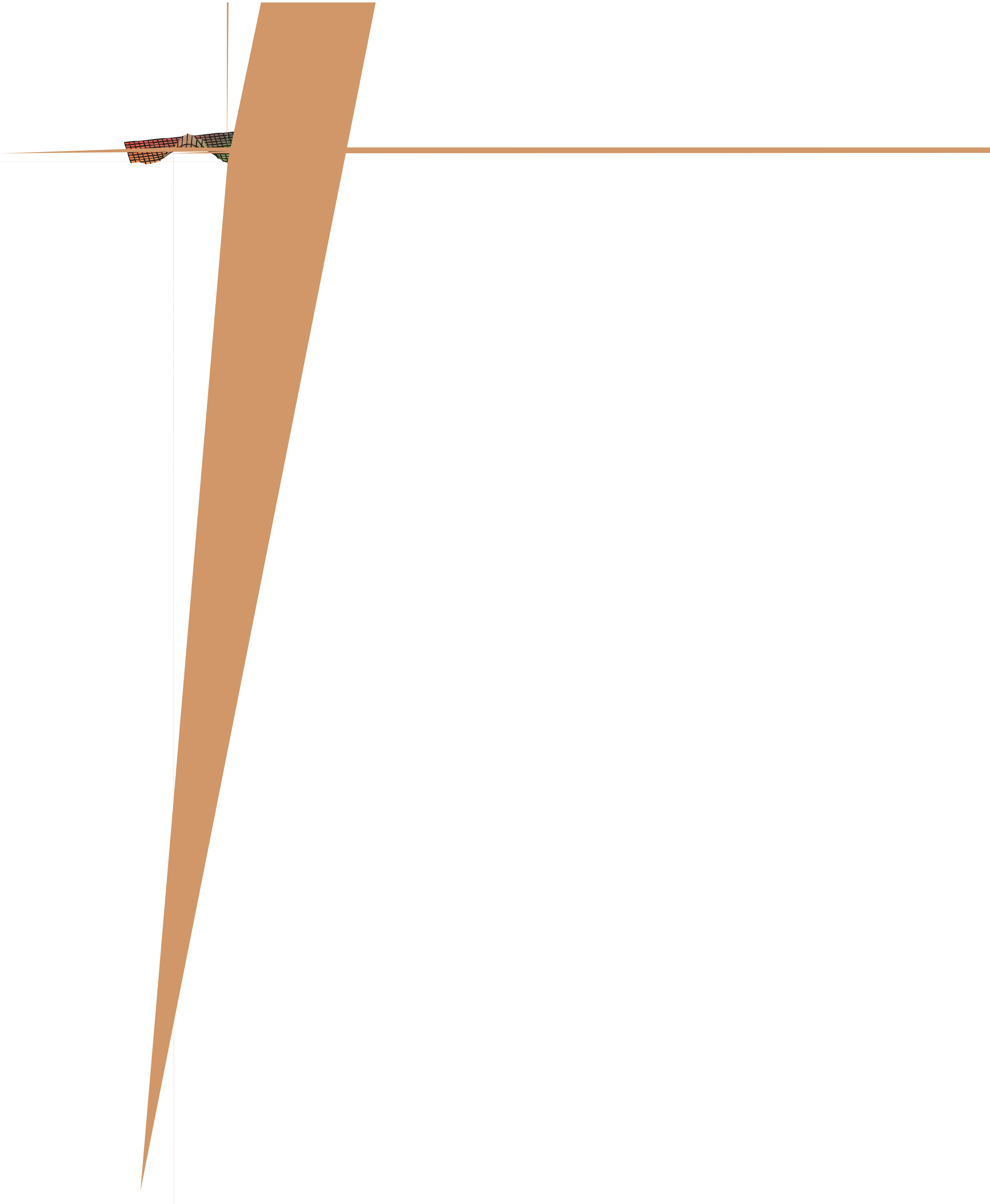
$H(_i \mid _i)$

$$H(_i \mid _i) = H(_i - _i) \equiv H(e_i) \quad (10)$$

, $H(_i) \equiv H(_i, _i)$ -

$$H(_i) = H(_i \mid _i) + H(_i) \equiv H(-e_i) + H(_i) = H(_i \mid _i) + H(_i) \equiv H(e_i) + H(_i) \quad (11)$$

e_i - ; $_i$ $_i$



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$ij \neq 0$.

$Q(\ \ _i) > 0$ **R** ,

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P **D**. □

Corollary 1: Q

$$Q(\ _1 + \ _2) = Q(\ _1) + Q(\ _2) \qquad (19)$$

$_1$ $_2$; (17)

; $Q(\ _j) > 0$

$_j$.

Proof: ? (19)

. Q , (19)

$$Q^2(\ _1 + \ _2) = Q^2(\ _1) + Q^2(\ _2) + 2Q(\ _1)Q(\ _2) \geq Q^2(\ _1) + Q^2(\ _2),$$

$Q(\ _1) = Q(\ _2) = 0$.

2. □

2 1 *sufficient*

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(> 2)

not $\text{cum}_k(\ _1 + \ _2) = \text{cum}_k(\ _1) + \text{cum}_k(\ _2)$

$_1$ $_2$.

. , $_1 = |\ _1|e^{j\theta_1}$, $_2 = |\ _2|e^{j\theta_2}$ $(\theta_1, \theta_2 \in \mathbb{R})$,

$$_1 + \ _2 = |\ _1|e^{j\theta_1} + |\ _2|e^{j\theta_2} = \sqrt{|\ _1|^2 + |\ _2|^2 + 2|\ _1||\ _2|\cos(\theta_1 - \theta_2)}e^{j\theta},$$

$\theta = \arctan \frac{_1 \quad \theta_1 + \ _2 \quad \theta_2}{_1 \quad \theta_1 + \ _2 \quad \theta_2}$. , $\theta_1 - \theta_2$, $||\ _1| - |\ _2|| \leq |\ _1 + \ _2| \leq |\ _1| + |\ _2|$.

Theorem 3: Q (2-)

; $= |\ |e^{j\theta}$ $(\theta \in \mathbb{R})$,

$\overline{Q}(\ ) = Q(|\ |) = \sqrt{Q^2(|\ |\cos\theta) + Q^2(|\ |\sin\theta)}$

(2-)

.

Proof: . $_1 = |\ _1|e^{j\theta_1}$

$_2 = |\ _2|e^{j\theta_2}$

; $= |\ |e^{j\theta} = \ _1 + \ _2$, $=$

$_1 + \ _2$ $= \ _1 + \ _2$;

Q ,

$$Q^2(|\ |\cos\theta) \geq Q^2(|\ _1|\cos\theta_1) + Q^2(|\ _2|\cos\theta_2)$$
$$Q^2(|\ |\sin\theta) \geq Q^2(|\ _1|\sin\theta_1) + Q^2(|\ _2|\sin\theta_2).$$

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$$\overline{Q}^2(\ ) = Q^2(|\ |) = Q^2(|\ |\cos\theta) + Q^2(|\ |\sin\theta) \geq Q^2(|\ _1|) + Q^2(|\ _2|) = \overline{Q}^2(\ _1) + \overline{Q}^2(\ _2), \qquad (20)$$

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Lemma 1: $Q(\ )$ $(\ \in \mathbb{C})$ -

; $\tilde{\mathbf{z}} = [\ \ ,\ \ ]$, Q

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$\tilde{\mathbf{z}} \in \mathbb{R}^2$; , ( ) :
$Q(\boldsymbol{\alpha} + \tilde{\mathbf{z}}) = Q(\tilde{\mathbf{z}})$ $(\forall \boldsymbol{\alpha} \in \mathbb{R}^2)$; ( ) :
$Q(\alpha\tilde{\mathbf{z}}) = |\alpha|Q(\tilde{\mathbf{z}})$ $(\forall \alpha \in \mathbb{R})$.

Proof: .

. $\alpha =$

$|\alpha|e^{j0}$ (. .,). ? . □

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Theorem 4: , Q -

≥ 2,

$-\sum_{i\ \ 1}^{n} Q^k(\ _i)$ $-\sum_{i\ \ 1}^{n} Q^{2k}(\ _i)$

;

> 2 $\{\ _j\}$,

$Q(\ _j) = 0$.

Proof: 15

. $= 2$ $\mathbf{R} = \mathbf{WA}$, -

Q, $Q^2(\ _i) \leq \sum_{j\ \ 1}^{n} |\ _{ij}|^2 Q^2(\ _j) \leq \sum_{j\ \ 1}^{n} Q^2(\ _j)$ $|\ _{ij}| \leq \sum_{k\ \ 1}^{n} |\ _{ik}|^2 = 1$; > 2, $Q^k(\ _i) \leq \sum_{j\ \ 1}^{n} |\ _{ij}|^k Q^k(\ _j) \leq \sum_{j\ \ 1}^{n} Q^k(\ _j)$.

$-\sum_{i\ \ 1}^{n} Q^k(\ _i) \geq -\sum_{j\ \ 1}^{n} Q^k(\ _j)$,

$-\sum_{i\ \ 1}^{n} Q^k(\ _i)$.

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Remark:

$\mathbf{0}$: $\mathbb{E}[\mathbf{x}\mathbf{x}^H] = \mathbf{A}\mathbb{E}[\mathbf{s}\mathbf{s}^H]\mathbf{A}^H = \mathbf{U}\boldsymbol{\Sigma}\mathbf{U}^H$, Σ

- , $\mathbf{U}$

. $\mathbf{z} = \mathbf{D}^{-1/2}\mathbf{U}^H\mathbf{x} = \mathbf{D}^{-1/2}\mathbf{U}^H\mathbf{A}\mathbf{s} \equiv \tilde{\mathbf{A}}\mathbf{s}$, $\tilde{\mathbf{A}} = \mathbf{D}^{-1/2}\mathbf{U}^H\mathbf{A}$

$\mathbb{E}[\mathbf{z}\mathbf{z}^H] = \tilde{\mathbf{A}}\mathbb{E}[\mathbf{s}\mathbf{s}^H]\tilde{\mathbf{A}}^H = \mathbf{I}$;

$\tilde{\mathbf{A}}$ $\mathbb{E}[\mathbf{s}\mathbf{s}^H] = \mathbf{I}$.

B. Examples of Contrast Functions

1) Range Function: 15 21 ,

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, $R(\ )$,

:

$$R(\ ) = d, \qquad \{\ (\ ) \geq 0 \ \big|\ |\ -\ _0| \leq d\} \tag{21}$$

$_0$,

$d \in \mathbb{R}^+$.

Theorem 5: -

$_1$ $_2$,

$$R(\ _1 +\ _2) = R(\ _1) + R(\ _2). \tag{22}$$

Proof: $|\ _1 +\ _2| \leq |\ _1| + |\ _2|$. □

Corollary 2: $= \alpha\ _1 + \beta\ _2$, $_1$ $_2$

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$\alpha, \beta \in \mathbb{C}$,

$$R(\ ) = |\alpha| \cdot R(\ _1) + |\beta| \cdot R(\ _2). \tag{23}$$

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$\alpha\ _1$ ($\beta\ _2$) $|\alpha|$ (

$|\beta|$) $_1$ ($_2$),

α (β).

$Q(\ _i) = R(\ _i)$, (17)

$$J(\mathbf{W}) = \sum_{i\ \ 1}^{n} \log\Big(\sum_{j\ \ 1}^{n} |\ _{ij}| R(\ _j)\Big) - \log|\ \ (\mathbf{W})|. \tag{24}$$

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Definition 6:

, $\{\ : |\ | \leq +\infty\}$;

$\int_0^d\ (|\ |) d|\ | \approx 1 \quad (0 < d < +\infty)$.

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2) Shannon Entropy Function: A -

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, $H(\ _i)$.

$$\overline{Q}(\ _i) = Q(|\ _i|) = \exp(H(|\ _i|)), \tag{25}$$

entropy power inequality, (25)

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(25) (17) (6)

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$\{\ _i\}$, $\{|\ _i|\}$

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16 , 17 .

3) Rényi Entropy Function: -

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$(0 <\ \ \in \mathbb{R})$:

$$H_k(\ _i) = \frac{1}{1 -\ } \log\Big(\int\ (\ _i)^k d\ _i\Big). \tag{26}$$

$\to 1$,

. $= 2$,

2 *extension entropy*

$$H_2(\ _i) = -\log\Big(\int\ (\ _i)^2 d\ _i\Big).$$

Jensen inequality, $H_2(\ _i) \leq H(\ _i)$.

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$H_k(\ _i) \geq H_r(\ _i)$ $>$.

Lemma 2: $\{\ :\ (\ ) = g(|\ |)\}$, ,

$\exp(H_2(\ ))$, .

Proof: , $H_2(\ ) = H_2(\ + \alpha)$

α. ? ,

$$\begin{aligned} H_2(\alpha\ ) &= -\log\Big(\int g(|\alpha\ |)^2 d|\alpha\ |\Big) \\ &= -\log\Big(\int g(|\ |)^2 d|\ |\Big) + \log|\alpha|. \end{aligned}$$

$\exp(H_2(\alpha\ )) = |\alpha| \exp(H_2(\ ))$. ,

$\exp(H_2(\ ))$. □

, $\overline{Q}(\ _i) = Q(|\ _i|) = \exp(H_2(|\ _i|)) = \frac{1}{\int p(|y_i|)^2 dy_i}$,

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26,27 ; $\overline{Q}$

. $\{|\ _i|\}$,

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4) Fisher Information Function:

$$\mathbf{G} = \mathbb{E}[\psi(\ )^2] = \mathbb{E}\Big[\Big(\frac{d \log\ (\ )}{d}\Big)^2\Big], \tag{27}$$

$\psi(\ ) = \frac{d\ \ \ p(\ )}{d} = \frac{p'(\ )}{p(\ )}$.

$Q(\ ) = \mathbf{G}^{-1/2}$, Q

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$$\overline{Q}(\ ) = \mathbb{E}\Big[\Big(\frac{\ '(|\ |)}{(|\ |)}\Big)^2\Big]^{-1/2}.$$

- , -

$\psi_k(\ ) =$

$$Q(\ ) = \quad (\Sigma\ )^{1/2} \equiv \quad \Big(\mathrm{cov}[\tilde{\mathbf{z}}]\Big)^{1/2},$$

$$\begin{aligned} Q(\omega\ ) &= \quad \Big(\mathrm{cov}[\tilde{\mathbf{z}}']\Big)^{1/2} = \quad \Big(\mathrm{cov}[\Omega\tilde{\mathbf{z}}]\Big)^{1/2} \\ &= \quad \Big(\Omega\mathrm{cov}[\tilde{\mathbf{z}}]\Big)^{1/2} = \quad (\Omega)^{1/2} \quad \Big(\mathrm{cov}[\tilde{\mathbf{z}}]\Big)^{1/2} \\ &= (\omega^2\ + \omega^2\ )^{1/2} \quad (\Sigma\ )^{1/2} = |\omega| \quad (\Sigma\ )^{1/2}. \end{aligned}$$

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$Q(\ )$.

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$$(\ ) \approx \mathcal{N}(\ )\left(1 + \frac{\kappa^3}{}\mathcal{H}_3(\ ) + \frac{\kappa^4}{2}\mathcal{H}(\ )\right)($$