

# The fee-ea-ad-eKell-c-e

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**Abstract** We propose an evolutionary framework for optimal portfolio growth theory in which investors subject to environmental pressures allocate their wealth between two assets. By considering both absolute wealth and relative wealth between investors, we show that different investor behaviors survive in different environments. When investors maximize their relative wealth, the Kell criterion is optimal only under certain conditions, which are identified. The initial relative wealth plays a critical role in determining the deviation of optimal behavior from the Kell criterion regardless of whether the investor is myopic across a single time period or maximizing wealth over an infinite horizon. We relate these results to population genetics, and discuss testable consequences of these findings using experimental evolution.

**Keywords** Kell criterion · Portfolio optimization · Adaptive Markets Hypothesis · Evolutionary game theory

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## 1 I d c

The allocation of wealth among alternative assets is one of an individual's most important financial decisions. The groundbreaking work of [Markowitz \(1952\)](#) in mean-variance theory, used to analyze asset allocation, has remained the cornerstone of modern portfolio theory. This has led to numerous breakthroughs in financial economics, including the famous Capital Asset Pricing Model ([Sharpe 1964](#); [Lintner 1965a, b](#); [Treynor 1965](#); [Mossin 1966](#)). The influence of this paradigm goes far beyond academia, however. For example, it has become an integral part of modern investment management practice ([Reilly and Brown 2011](#)).

More recently, economists have also adopted the use of evolutionary principles to understand economic behavior, leading to the development of evolutionary game theory ([Marschall and Smith 1982](#)); the evolutionary implications of probability matching ([Cooper and Kaplan 1982](#)), group selection ([Zhang et al. 2014a](#)), and cooperation and altruism ([Alexander 1974](#); [Hirshleifer 1977, 1978](#)); and the process of selection of firms ([Luo 1995](#)) and traders ([Blume and Easley 1992](#); [Kogan et al. 2006a](#); [Hirshleifer and Teoh 2009](#)). The Adaptive Markets Hypothesis ([Lo 2004, 2017](#)) provides economics with a more general evolutionary perspective, reconciling economic theories based on the Efficient Markets Hypothesis with behavioral economics. Under this hypothesis, the neoclassical models of rational behavior coexist with behavioral models, and what had previously been cited as counterexamples to rationality—loss aversion, overconfidence, overreaction, and other behavioral biases—become consistent with an evolutionary model of human behavior.

The evolutionary perspective brings new insights to economics from beyond the traditional neoclassical realm, helping to reconcile inconsistencies in behavior between *Homo economicus* and *Homo sapiens* ([Kahneman and Tversky 1979](#); [Brennan and Lo 2011](#)). In particular, evolutionary models of behavior provide important insights into the biological origin of time preference and utility functions ([Rogers 1994](#); [Waldman 1994](#); [Samuelson 2001](#); [Zhang et al. 2014b](#)), in fact justifying their existence, and allow us to derive conditions about their functional form ([Hansson and Stuart 1990](#); [Robson 1996, 2001a](#)) (see [Robson 2001b](#); [Robson and Samuelson 2009](#) for comprehensive reviews of this literature). In addition, the experimental evolution of biological organisms has been suggested as a novel approach to understanding economic preferences, given that it allows the empirical study of preferences by placing organisms in specifically designed environments ([Burnham et al. 2015](#)).

The evolutionary approach to investing is closely related to optimal portfolio growth theory, as explored by [Kell \(1956\)](#), [Hakansson \(1970\)](#), [Thorp \(1971\)](#), [Algoet and Cover \(1988\)](#), [Browne and Whitt \(1996\)](#), and [Aurell et al. \(2000\)](#), among others. While the evolutionary framework tends to focus on the long-term performance of a strategy, investors are also concerned with the short to medium term ([Browne 1999](#)). Mopic investor behavior has been documented in both theoretical and empirical studies ([Strot 1955](#); [Strotman 1983](#); [Thaler et al. 1997](#); [Bushee 1998](#)). Since much of the field of population genetics focuses on short-term competition between different

types of individuals, population geneticists have applied their ideas to portfolio theory (Frank 1990, 2011; Orr 2017), in some cases considering the maximization of one-period expected wealth.

The market dynamics of investment strategies under evolutionary selection have been explored under the assumption that investors will try to maximize absolute wealth (Evstigneev et al. 2002; Amir et al. 2005; Hens and Schenk-Hopp 2005; Evstigneev et al. 2006). Some studies have found that individual investors with more accurate beliefs will accumulate more wealth, and thus dominate the economy (Sandroni 2000, 2005), while others have argued that wealth dynamics need not lead to rules that maximize expected utility using rational expectations (Blume and Easley 1992), and that investors with incorrect beliefs may drive out those with correct beliefs (Blume and Easley 2006). Research on the performance of rational versus irrational traders has also adopted evolutionary ideas; for example, it has been shown that irrational traders can survive in the long run, resulting in prices that diverge from fundamental values (Long et al. 1990, 1991; Biais and Shadrur 2000; Hirshleifer and Luo 2001; Hirshleifer et al. 2006; Kogan et al. 2006b; Yan 2008).

Traditional portfolio growth theory has focused on absolute wealth and the Kell criterion (Kell 1956; Thorp 1971). Instead of studying the growth of absolute wealth, however, we will consider the relative wealth in the spirit of Orr (2017). Relative wealth or income has been discussed in a number of studies (Robson 1992; Bakshi and Chen 1996; Corneo and Jeanne 1997; Hens and Schenk-Hopp 2005; Frank 2011), and the behavioral economics literature provides voluminous evidence that investors sometimes assess their performance relative to a reference group (Frank 1985; Clark and Oswald 1996; Clark et al. 2008). It is particularly important to understand the consequences of investment decisions in a setting where relative wealth is the standard, not absolute wealth. As Burnham et al. (2015) pointed out, if people are envious but caring about relative wealth, then free trade may make all parties richer, but may cause envious people to be less happy. If economics misunderstands human nature, then free trade may simultaneously increase wealth and unhappiness.

In this paper, we compare the implications of maximizing relative wealth to maximizing absolute wealth over both short-term and long-term investment horizons. We use ideas from Orr (2017), and compare his results to an extension of the binary choice model of Brennan and Lo (2011). We consider two assets in a discrete-time model, and an investor who allocates her wealth between the two assets. Rather than maximizing her absolute wealth, the investor maximizes her wealth relative to another investor with a fixed behavior. We consider the cases of one time period, multiple periods, and an infinite time horizon. We then ask the question: what is the optimal behavior for an investor as a function of the environment, given that the environment consists of the asset returns and the behavior of the other participants? In our approach, we define relative wealth as a proportion of the total wealth, which corresponds closely to the allele frequency in population genetics. This analogy acts as a bridge to earlier literature on the relevance of relative wealth to behavior. While some of our results will be familiar to population geneticists, they do not appear to be widely known in a financial context. For completeness, we derive them from first principles in this new context.

Our approach leads to several interesting conclusions about the Kell criterion. We show that it is the optimal behavior if the investor maximizes her absolute wealth in the

case of an infinite horizon (see also [Brennan and Lo 2011](#)). In the case that the investor maximizes her relative wealth, we identify the conditions under which it is optimal, and the conditions under which the investor should deviate from it. The investor's initial relative wealth—which represents the investor's market power—plays a critical role. Moreover, the dominant investor's optimal behavior is different from the minorant investor's optimal behavior.

In Sect. 2 of this paper, we consider a two-asset model in which investors maximize their absolute wealth. It is shown that the long-run optimal behavior is equivalent to the behavior implied by the Kell criterion. Section 3 extends the binary choice model, and considers in a non-game-theoretic framework the case of two investors who maximize their wealth relative to the population, given the other investor's behavior. The Kell criterion emerges as a special case under certain environmental conditions. Section 4 provides a numerical example to illustrate the theoretical results. Section 5 discusses several implications which can be tested through experimental evolutionary techniques. We end with a discussion in Sect. 6, and provide proofs in Appendix A.

## 2 Main Results: the Kell Criterion

Consider two assets,  $a$  and  $b$ , in a discrete-time model, each generating gross returns  $X_a \in (0, \infty)$  and  $X_b \in (0, \infty)$  in one period. For example, asset  $a$  can be a risky asset whereas asset  $b$  can be the riskless asset. In this case,  $X_a \in (0, \infty)$  and  $X_b = 1 + r$ , where  $r$  is the risk-free interest rate. In general,  $(X_{a,t}, X_{b,t})$  are IID over time  $t = 1, 2, \dots$ , and are described by the probability distribution function  $\Phi(X_a, X_b)$ .

Consider an investor who allocates  $f \in [0, 1]$  of her wealth in asset  $a$  and  $1 - f$  in asset  $b$ . We will refer to  $f$  as the investor's behavior henceforth. We assume that:

**A1**  $(X_a, X_b)$  and  $\log(fX_a + (1 - f)X_b)$  have finite moments up to order 2 for all  $f \in [0, 1]$ .

Note that Assumption 1 guarantees that the gross return of an investment portfolio is positive. In other words, the investor cannot lose more than what she has. This is made possible by assuming that  $X_a$  and  $X_b$  are positive and  $f$  is between 0 and 1. In other words, the investor only allocates her money between two assets, and no short-selling or leverage is allowed.<sup>1</sup>

Let  $n_t^f$  be the total wealth of investor  $f$  in period  $t$ . To simplify notation, let  $\omega_t^f = fX_{a,t} + (1 - f)X_{b,t}$  be the gross return of investor  $f$ 's portfolio in period  $t$ . With these notational conventions in mind, the portfolio growth from period  $t - 1$  to period  $t$  is:

$$n_t^f = n_{t-1}^f (fX_{a,t} + (1 - f)X_{b,t}) = n_{t-1}^f \omega_t^f.$$

<sup>1</sup> One could relax this assumption by allowing short-selling and leverage, which corresponds to  $f < 0$  or  $f > 0$ . However,  $f$  still needs to be restricted such that  $fX_a + (1 - f)X_b$  is always positive. This does not change our results in an essential way, but it will complicate the presentation of some results mathematically. Therefore, we stick to the simple assumption that  $f \in [0, 1]$  as in [Brennan and Lo \(2011\)](#).

Through backward recursion, the total wealth of investor  $f$  in period  $T$  is given b

$$n_T^f = \prod_{t=1}^T \omega_t^f = \exp \left( \sum_{t=1}^T \log \omega_t^f \right).$$

Taking the logarithm of wealth and applying Kolmogorov's law of large numbers, we have:

$$\frac{1}{T} \log n_T^f = \frac{1}{T} \sum_{t=1}^T \log \omega_t^f \xrightarrow{P} \mathbb{E} \left[ \log \omega_t^f \right] = \mathbb{E} \left[ \log (f X_a + (1 - f) X_b) \right], \quad (1)$$

as  $T$  increases without bound, where  $\xrightarrow{P}$  in (1) denotes convergence in probability. We have assumed that  $n_0^f = 1$  without loss of generality.

The expression (1) is simply the expectation of the log-geometric-average growth rate of investor  $f$ 's wealth, and we will call it  $\mu(f)$  henceforth:

$$\mu(f) = \mathbb{E} \left[ \log (f X_a + (1 - f) X_b) \right]. \quad (2)$$

The optimal  $f$  that maximizes (2) is given b

**P 1** *The optimal allocation  $f^{Kelly}$  that maximizes investor  $f$ 's absolute wealth as  $T$  increases without bound is*

$$f^{Kelly} = \begin{cases} 1 & \text{if } \mathbb{E}[X_a/X_b] > 1 \text{ and } \mathbb{E}[X_b/X_a] < 1, \\ \text{solution to (4)} & \text{if } \mathbb{E}[X_a/X_b] \geq 1 \text{ and } \mathbb{E}[X_b/X_a] \geq 1, \\ 0 & \text{if } \mathbb{E}[X_a/X_b] < 1 \text{ and } \mathbb{E}[X_b/X_a] > 1, \end{cases} \quad (3)$$

where  $f^{Kelly}$  is defined implicitly in the second case of (3) by:

$$\mathbb{E} \left[ \frac{X_a - X_b}{f^{Kelly} X_a + (1 - f^{Kelly}) X_b} \right] = 0. \quad (4)$$

The optimal allocation given in Proposition 1 coincides with the Kelly criterion (Kelly 1956; Thorp 1971) in probability theory and the portfolio choice literature. To emphasize this connection, we refer to this optimal allocation as the Kelly criterion henceforth. As we will see, in the case of maximizing an individual's relative wealth, the Kelly criterion plays a key role as a reference strategy.

In portfolio theory, the Kelly criterion is used to determine the optimal size of a series of bets in the long run. Although this strategy's promise of doing better than any other strategy in the long run seems compelling, some researchers have argued against it, principally because the specific investing constraints of an individual may override

### 3 Mathematical Preliminaries

In this section, we consider two investors. The first investor allocates  $f \in [0, 1]$  of her wealth in asset  $a$  and  $1 - f$  in asset  $b$ . The second investor allocates  $g \in [0, 1]$  of his wealth in asset  $a$  and  $1 - g$  in asset  $b$ . Investor  $f$ 's objective is to maximize the proportion of her wealth relative to the total wealth in the population, which we define as investor  $f$ 's relative wealth. Note that we use  $f$  and  $g$  to mean the proportion of wealth and as a label for the investor, to simplify notation.

Here we can introduce a concept taken from evolutionary theory. In population genetics, the metric for natural selection is the expected reproduction of a genotype divided by the average reproduction of the population, i.e., the relative reproduction, analogous to investor  $f$ 's relative wealth. Our consideration of the relative wealth rather than the absolute wealth naturally unlocks existing tools and ideas from population genetics for us.

In the case of maximizing relative wealth, the initial wealth plays an important role in the optimal allocation. Let  $\lambda \in (0, 1)$  be the relative initial wealth of investor  $f$ :

$$\lambda = \frac{n_0^f}{n_0^f + n_0^g}.$$

Let  $q_t^f$  be the relative wealth of investor  $f$  in subsequent periods  $t = 1, 2, \dots$ .  $q_t^f$  and  $q_t^g$  are defined similarly:

$$q_t^f = \frac{n_t^f}{n_t^f + n_t^g} = \frac{1}{1 + n_t^g/n_t^f},$$

$$q_t^g = 1 - q_t^f.$$

It is obvious that the ratio  $n_t^g/n_t^f$  is sufficient to determine the relative wealth  $q_t^f$ . Let  $R_T^f$  be the  $T$ -period average log-relative-growth:

$$R_T^f = \frac{1}{T} \log \frac{\prod_{t=1}^T \omega_t^g}{\prod_{t=1}^T \omega_t^f} = \frac{1}{T} \sum_{t=1}^T \log \frac{\omega_t^g}{\omega_t^f}. \tag{5}$$

Then we can write the relative wealth in period  $T$  as:

$$q_T^f = \frac{1}{1 + \frac{n_T^g}{n_T^f}} = \frac{1}{1 + \frac{(1-\lambda) \prod_{t=1}^T \omega_t^g}{\lambda \prod_{t=1}^T \omega_t^f}} = \frac{1}{1 + \frac{1-\lambda}{\lambda} \exp\left(T R_T^f\right)}. \tag{6}$$

Analogous to Eqs. (5)–(6) are well known in the population genetics literature, used when the fitnesses of genotypes are assumed to vary randomly through time. (For reviews of this literature, see [Felsenstein 1976](#) and [Gillespie 1991](#), Chap. 4.)

### 3.1 Objective

We first consider a myopic investor, who maximises her expected relative wealth in the first period. By (6), the expectation of  $q_1^f$  is:

$$\mathbb{E}\left[q_1^f\right] = \mathbb{E}\left[\frac{1}{1 + \frac{1-\lambda}{\lambda} \frac{\omega^g}{\omega^f}}\right].$$

Here we have dropped the subscripts in  $\omega_1^f$  and  $\omega_1^g$ , and instead simply use  $\omega^f$  and  $\omega^g$ , because there is only one period to consider.

Given investor  $g$ , we denote  $f_1^*$  as investor  $f$ 's optimal allocation that maximises  $\mathbb{E}[q_1^f]$ . There is no general formula to compute  $\mathbb{E}[q_1^f]$  because it involves the expectation of the ratio of random variables. Population geneticists sometimes use diffusion approximations to estimate similar quantities, for example, the change in allele frequency (Gillespie 1977; Frank and Slatkin 1990; Frank 2011), which are essentially linear approximations of the nonlinear quantity using the Taylor series. The diffusion approximation is also used by Orr (2017) in a similar model for relative wealth.

Without the diffusion approximation, one can still characterise  $f$

than investor  $g$ . Similarly, when investor  $g$  takes a position that is more conservative than the Kelly criterion, investor  $f$  should never be even more conservative than investor  $g$ .

It is interesting to compare the optimal behavior  $f_1^*$  with the Kelly criterion  $f^{Kelly}$ , which is provided in the next proposition. It shows that when  $g$  is not far from the Kelly criterion, the relationship between  $f_1^*$  and  $f^{Kelly}$  depends on the initial relative wealth of investor  $f$ .

**P 4** *If investor  $f$  is the dominant investor ( $\lambda > \frac{1}{2}$ ), then she should be locally more/less risky than Kelly in the same way as investor  $g$ : for small  $\epsilon > 0$ ,*

$$\begin{aligned} g = f^{Kelly} - \epsilon &\Rightarrow f_1^* < f^{Kelly}, \\ g = f^{Kelly} + \epsilon &\Rightarrow f_1^* > f^{Kelly}. \end{aligned}$$

*If investor  $f$  is the minorant investor ( $\lambda < \frac{1}{2}$ ), then she should be locally more/less risky than Kelly in the opposite way as investor  $g$ : for small  $\epsilon > 0$ ,*

$$\begin{aligned} g = f^{Kelly} - \epsilon &\Rightarrow f_1^* > f^{Kelly}, \\ g = f^{Kelly} + \epsilon &\Rightarrow f_1^* < f^{Kelly}. \end{aligned}$$

*If investor  $f$  starts with the same amount of wealth as investor  $g$  ( $\lambda = \frac{1}{2}$ ), then she should be locally Kelly:*

$$g \approx f^{Kelly} \Rightarrow f_1^* \approx f^{Kelly}.$$

Note that when  $g$  is far from the Kelly criterion, the conclusions in Proposition 4 may not necessarily hold. Section 4 provides a numerical example (see Fig. 1b) where investor  $f$  is the minorant investor ( $\lambda < \frac{1}{2}$ ) and  $g \ll f^{Kelly}$ , but  $f_1^* < f^{Kelly}$ . However, Orr (2017) has shown that these results are still approximately true for an  $g$  up to a diffusion approximation, which is consistent with the numerical results for maximizing one-period relative wealth in Fig. 1a. We will provide more discussion on this point in Sect. 4.

### 3.2 More details

The previous results are based on maximizing the expected relative wealth in period 1:  $\mathbb{E}[q_1^f]$ . To generalize these results to maximizing expected relative wealth in period  $T$ :  $\mathbb{E}[q_T^f]$ , we have:

**P 5** *The optimal behavior of investor  $f$  that maximizes expected relative wealth in the  $T$ th period is given by:*



$$f_T^* = \begin{cases} 1 & \text{if } \mathbb{E} \left[ \frac{\exp(TR_T^1) \left( T - \sum_{t=1}^T \frac{X_{bt}}{X_{at}} \right)}{\left( 1 + \frac{1-\lambda}{\lambda} \exp(TR_T^1) \right)^2} \right] > 0, \\ \text{solution to (10)} & \text{if } \mathbb{E} \left[ \frac{\exp(TR_T^1) \left( T - \sum_{t=1}^T \frac{X_{bt}}{X_{at}} \right)}{\left( 1 + \frac{1-\lambda}{\lambda} \exp(TR_T^1) \right)^2} \right] \leq 0 \text{ and } \mathbb{E} \left[ \frac{\exp(TR_T^0) \left( \sum_{t=1}^T \frac{X_{at}}{X_{bt}} - T \right)}{\left( 1 + \frac{1-\lambda}{\lambda} \exp(TR_T^0) \right)^2} \right] \geq 0, \\ 0 & \text{if } \mathbb{E} \left[ \frac{\exp(TR_T^0) \left( \sum_{t=1}^T \frac{X_{at}}{X_{bt}} - T \right)}{\left( 1 + \frac{1-\lambda}{\lambda} \exp(TR_T^0) \right)^2} \right] < 0, \end{cases} \quad (9)$$

where  $f_1^*$  is defined implicitly in the second case of (9) by:

$$\mathbb{E} \left[ \exp(TR_T^f) \sum_{t=1}^T \frac{X_{at} - X_{bt}}{fX_{at} + (1-f)X} \right]$$

of Sect. 4 show that the condition that  $g$  is near the Kell criterion is essential (see Fig. 2a).

**3.3 I e**

Recall from (5) that the  $T$ -period average log-relative-growth  $R_T^f$  is given b :

$$R_T^f = \frac{1}{T} \sum_{t=1}^T \log \omega_t^g - \frac{1}{T} \sum_{t=1}^T \log \omega_t^f \xrightarrow{P} \mu(g) - \mu(f), \tag{11}$$

as  $T$  increases without bound. It is therefore eas to see from (6) that:

**P 7** *As  $T$  increases without bound, the relative wealth of investor  $f$  converges in probability to a constant:*

$$q_T^f \xrightarrow{P} \begin{cases} 0 & \text{if } \mu(f) < \mu(g), \\ \lambda & \text{if } \mu(f) = \mu(g), \\ 1 & \text{if } \mu(f) > \mu(g). \end{cases} \tag{12}$$

Proposition 7 is consistent with well-known results in the population genetics literature (see Gillespie 1973, for example) as well as in the behavioral finance literature, as in Brennan and Lo (2011). It asserts that investor  $f$ ’s relative wealth will converge to 1 as long as its log-geometric-average growth rate  $\mu(f)$  is greater than investor  $g$ ’s. This implies that when  $T$  increases without bound, there are multiple behaviors that are all optimal in the following sense:

$$\begin{aligned} \arg \max_f \lim_{T \rightarrow \infty} q_T^f &= \arg \max_f \mathbb{E} \left[ \lim_{T \rightarrow \infty} q_T^f \right] \\ &= \arg \max_f \lim_{T \rightarrow \infty} \mathbb{E} \left[ q_T^f \right] = \{f : \mu(f) > \mu(g)\}. \end{aligned}$$

Note that the above equalit uses the dominant convergence theorem ( $q_T^f$  is alwa s bounded) to switch the limit and the expectation operator.

However, this is not equivalent to the limit of the optimal behavior  $f_T^*$  as  $T$  increases without bound because one cannot switch the operator,  $\arg \max$  and,  $\lim$  in general, and

$$\arg \max_f \lim_{T \rightarrow \infty} \mathbb{E} \left[ q_T^f \right] \neq \lim_{T \rightarrow \infty} \arg \max_f \mathbb{E} \left[ q_T^f \right].$$

In fact, Sect. 4 provides such an example.

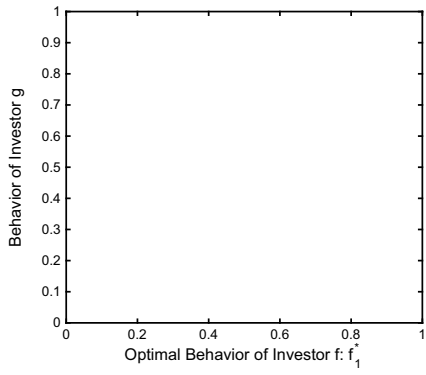
**4 A e ca e a e**

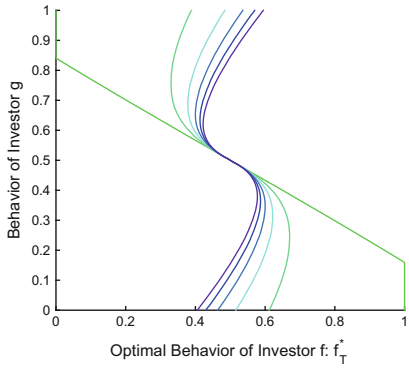
We construct a numerical example in this section to illustrate the results of Sects. 2 and 3. Consider the following two simple assets:

$$X_a = \begin{cases} \alpha & \text{with probability } p, \\ \beta & \text{with probability } 1 - p, \end{cases} \quad X_b = \gamma \quad \text{with probability } 1.$$

In this case, asset  $a$  is risky and asset  $b$  is riskless. The expected relative wealth of investor  $f$  in period  $T$  is explicitly given by :

$$\mathbb{E} \left[ q_T^f \right] = \sum_{k=0}^T \binom{T}{k} \quad (T)$$





rather than on the absolute wealth. Relative wealth is important financially because success and satisfaction are sometimes measured by investors relative to the success of others (Robson 1992; Frank 1990, 2011; Bakshi and Chen 1996; Clark and Oswald 1996; Corneo and Jeanne 1997; Clark et al. 2008). Our model considers the case of two investors in a non-game-theoretic framework. We show how the optimal behavior of one investor is dependent on the other investor's behavior, which might be far from the Kell criterion. While some of our results are already known in the finance literature or the population genetics literature, they are not known together in both, and therefore they are included for completeness.

We consider myopic investors who maximize their expected relative wealth over a single period, and investors who maximize their relative wealth over multiple periods. Similar consequences hold in both cases. When one investor is wealthier than the other, that investor should roughly mimic the other's behavior in being more or less aggressive than the Kell criterion. Conversely, when one investor is poorer than the other, that investor should roughly act in the opposite manner of the other investor (Orr 2017).

As described above, it should be possible to design empirical biological studies to test the ideas of this paper. For example, one could design an experimental evolutionary study with a riskless condition (with constant fitness, corresponding to a fixed payoff) and a risky condition (with variable fitness, corresponding to different payoffs), much like the numerical example considered in Sect. 4. More generally, one could design an experimental environment with two random fitnesses that follow two different distributions. By varying the proportion of each population type exposed to each environment, one could create any type of investor as described in our model. Eventually, one would observe the growth of different types of investors to test various predictions about relative wealth in this paper.

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## A Appendix A: Proof

*Proof of Proposition 1* See Brennan and Lo (2011). □

*Proof of Proposition 2* The first partial derivative of  $\mathbb{E}[q_1^f]$  to  $f$  is:

$$\frac{\partial \mathbb{E}[q_1^f]}{\partial f} = \lambda(1 - \lambda) \mathbb{E} \left[ \frac{(X_a - X_b)\omega^g}{(\lambda\omega^f + (1 - \lambda)\omega^g)^2} \right].$$

The second partial derivative of  $\mathbb{E}[q_1^f]$  to  $f$  is:

$$\frac{\partial^2 \mathbb{E}[q_1^f]}{\partial f^2} = -2\lambda^2(1 - \lambda) \mathbb{E} \left[ \frac{(X_a - X_b)^2 \omega^g}{(\lambda\omega^f + (1 - \lambda)\omega^g)^3} \right] \leq 0,$$

which indicates that  $\mathbb{E}[q_1^f]$  is a concave function of  $f$ . Therefore, it suffices to consider the value of the first partial derivative at its endpoints 0 and 1.

$$f_1^* = \begin{cases} 1 & \text{if } \frac{\partial \mathbb{E}[q_1^f]}{\partial f} \Big|_{f=1} > 0, \\ 0 & \text{if } \frac{\partial \mathbb{E}[q_1^f]}{\partial f} \Big|_{f=0} < 0, \\ \text{solution to } \frac{\partial \mathbb{E}[q_1^f]}{\partial f} = 0 & \text{otherwise.} \end{cases}$$

Proposition 2 follows from trivial simplifications of the above equation. □

*Proof of Proposition 3* Consider  $\frac{\partial \mathbb{E}[q_1^f]}{\partial f}$  when  $f = g$ :

$$\frac{\partial \mathbb{E}[q_1^f]}{\partial f} \Big|_{f=g} = \lambda(1 - \lambda) \mathbb{E} \left[ \frac{X_a - X_b}{fX_a + (1 - f)X_b} \right].$$

Note that the righthand side consists of a factor that also appears in the first order condition (4) of the Kell criterion. Therefore its sign is determined by whether  $f$  is larger than  $f^{Kelly}$ :

$$\frac{\partial \mathbb{E}[q_1^f]}{\partial f} \Big|_{f=g} \begin{cases} > 0 \text{ if } f = g < f^{Kelly}, \\ = 0 \text{ if } f = g = f^{Kelly}, \\ < 0 \text{ if } f = g > f^{Kelly}. \end{cases} \tag{A.1}$$

Since  $\mathbb{E}[q_1^f]$  is concave as a function of  $f$  for an  $g$ , we know that:

$$f_1^* \begin{cases} > g \text{ if } g < f^{Kelly}, \\ = g \text{ if } g = f^{Kelly}, \\ < g \text{ if } g > f^{Kelly}, \end{cases}$$

which completes the proof. □

*Proof of Proposition 4* The cross partial derivative of  $\mathbb{E}[q_1^f]$  is:

$$\frac{\partial^2 \mathbb{E}[q_1^f]}{\partial f \partial g} = \lambda(1 - \lambda) \mathbb{E} \left[ \frac{(X_a - X_b)^2 (\lambda \omega^f - (1 - \lambda) \omega^g)}{(\lambda \omega^f + (1 - \lambda) \omega^g)^3} \right].$$

Consider  $\frac{\partial^2 \mathbb{E}[q_1^f]}{\partial f \partial g}$  when  $f = g = f^{Kelly}$ :

$$\frac{\partial^2 \mathbb{E}[q_1^f]}{\partial f \partial g} \Big|_{f=g} = 2\lambda(1 - \lambda) \left( \lambda - \frac{1}{2} \right) \mathbb{E} \left[ \left( \frac{X_a - X_b}{fX_a + (1 - f)X_b} \right)^2 \right] \begin{cases} < 0 \text{ if } \lambda < \frac{1}{2}, \\ = 0 \text{ if } \lambda = \frac{1}{2}, \\ > 0 \text{ if } \lambda > \frac{1}{2}. \end{cases}$$

The first order condition (A.1) is 0 when  $f = g = f^{Kelly}$ , so when  $g$  is near  $f^{Kelly}$ , the sign of the first order condition is determined by whether  $\lambda$  is greater than, equal to, or

less than  $1/2$ . For example, if  $\lambda < 1/2$ , then the derivative of the first order condition (A.1) with respect to  $g$  is negative, which implies that the first order condition is negative when  $g = f^{Kelly} + \epsilon$ , where  $\epsilon$  is a small positive quantity. Therefore, when  $g = f^{Kelly} + \epsilon$ ,  $f_1^*$  is smaller than  $f^{Kelly}$ . The cases when  $\lambda > 1/2$  and  $\lambda = 1/2$  follow similarly.  $\square$

*Proof of Proposition 5* The first partial derivative of  $\mathbb{E}[q_T^f]$  to  $f$  is:

$$\frac{\partial \mathbb{E}[q_T^f]}{\partial f} = \frac{1 - \lambda}{\lambda} \mathbb{E} \left[ \frac{\exp(T R_T^f) \sum_{i=1}^T \frac{X_{at} - X_{bt}}{f X_{at} + (1-f) X_{bt}}}{\exp(T R_T^f) \sum_{i=1}^T \frac{X_{at} - X_{bt}}{f X_{at} + (1-f) X_{bt}}} \right]$$



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