Random walks, boundaries and measures in Conformal Dynamical System

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Fatou and Julia

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If $f\colon S \to S$ is conformal on a hyperbolic surface S, then the Fatou set is the whole surface $F(f) = S$.

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polynomial dynamics

Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a polynomial and $\mathcal{A}(\infty)$ be the superattracting basin of ∞. Its complement $K(f) = \widehat{\mathbb{C}} \setminus \mathcal{A}(\infty)$ is called the flled Julia set. By Böttcher's theorem (superattracting basin either is conformally conjugate to z^d or contains another critical point), there is a dichotomy

[redacted]

The filled Julia set $K(f)$ is connected iff its complement $\mathcal{A}(\infty)$ is conformally conjugate to the action of z^d on the unit disk $\mathbb{D}$.

polynomial dynamics

The closure of each (super)attracting basin contains the Julia set.
Hence $J(f) = \partial\mathcal{A}(\infty) = \partial K(f)$.

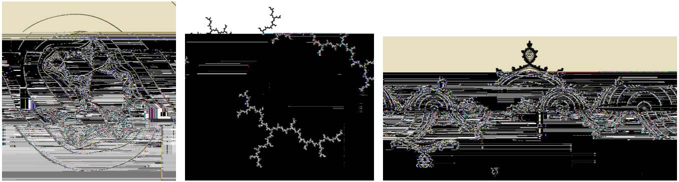

Figure: Connected Julia sets of polynomials $z^2 + (-0.1226 + 0.7449\mathbf{i})$, $z^2 + i$, and $z^2 - 1$.

harmonic measures for polynomials

We moreover assume that $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a *hyperbolic* polynomial with a connected Julia set $J = J(f)$. The basin of ∞ is also denoted by $\Omega = \mathcal{A}(\infty)$.
Hyperbolic: equipped with some conformal metric, $|f'(z)| > 1$ for all $z \in J(f)$. That is, f is expanding on the Julia set J.

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The harmonic measure $\{\nu_x\}_{x\in\Omega}$ is a family of Borel probabilities $\{\nu_x\} \subseteq \mathcal{B}(J)$ such that the following (Solution of Dirichlet problem) holds: for all continuous function $\phi\colon J \to$, the function

$$\widetilde{\phi}(x) := \int_{z\in J} \phi(z)d\nu_x(z), x \in \Omega, \tag{1.1}$$

is a harmonic extension of ϕ.

harmonic measures for polynomials

Consider a Riemann mapping $\phi\colon \mathbb{D} \to \Omega$ from the unit disk to the basin of ∞. The harmonic measure for $\mathbb{D}$ (seen from 0) is the Lebesgue measure $\nu_{0,\mathbb{D}} = \lambda$.

Recall: harmonicity is preserved by conformal maps. If ϕ extends continuously to $\partial\mathbb{D}$, then $\nu_{\phi(0),\Omega} = \phi_*\lambda$. It remains true in general by Fatou's theorem (angular limit of ϕ exists λ-a.e.)

ergodic properties of the harmonic measure: Brolin and Lyubich

The harmonic measure $\nu = \nu_\infty$ seen from ∞ is f-invariant and supported on the Julia set J. (By Böttcher's theorem, choose ϕ such that $f\phi(z) = \phi(z^d)$)

ergodic properties of the harmonic measure: Brolin and Lyubich

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The harmonic measure ν is the measure of maximal entropy.

Recall: variational principle for entropy:

$$h_\mu(f) \leqslant h_{\mathrm{top}}(f) = \ln(\deg f). \tag{1.2}$$

Motivation of our work: generalize the classical harmonic measure.

R *QM7Q`K H .vM KB+ H avbi2Kb M/ amHHB
Ai2` iBQM Q7 ` iBQM H K Tb
EH2BMB M ;`QmTb

k #QmM/ `v M/ K2 bm`2b
?vT2`#QHB+ ;`QmTb
1tT M/BM; /vM KB+b

. B b + ` 2 i 2 U } M B i 2 H v ; 2 M 2 ` i 2 / V U:i Q `7 b B Q M @ 7

A b Q(K j) = Conf(C) = S a(C, C) U + i B M a; = Q M = @H j V X

.Bb+`2i2 U}MBi2Hv ;2M2` i2/V U:iQ`7b BQM@7 Ab Q(K j) = Conf(C) = S a(C; C) U +iBMa k = QM = @H j V X h?2 HBKBi:b=2fi lim ;M t: ;M 2 :g a k X Aib +QKTH2K2

: = a k n : Bb + HH/2Q K?B2M Q7 /Bb+QQ M?i2B Q m/BM b2iX

T`QT2`iB2b	EH2BMB M ;`QmTb`	iBQM H K Tb
/B+?QiQKv, +? QiB+ pXbX	MHB`KKBHb2i pXbX	Q`CBmHBv b2i pXbX 6 iQm b
T ` K2i2`b	;2M2` iQ`b	+Q2{+B2MibUT`2BK ;2b
/2MbBiv Q7 2tT MbBQM2 BM HBKBi b/2iMb2 BM CmHB b2i BM +? QiB+ T	?vT2`#QHB+ }t2/2tQ BMiM ; T2`BQ/B+ TQB	
}MBi2M2bb Q7 MQ`K7HQ`Tb	}MBi2M2bb Q7 ?2QM2K`BM; /QK BM	
?vT2`#QHB+Biv	+QMp2t +Q+QKT 2tiT MbBQM QM	CmHB b
bi`m+im` H bi #BHBiv Q7 ?vT2`#QHB+Biv h`m2		h`m2
;2QK2i`Bx iBQM	MMQMöb +QMb2`mi`biQMöb	+? ` +i2`Bx i

h?2 bBKTH2bi + b2, ?vT2`#QHB+ ` iBQM H K
CmHB b2i- r2 Q#i BM M7 2Ci T OX/BM; K T

h?2 bBKTH2bi + b2, ?vT2`#QHB+ ` iBQM H K
CmHB b2i- r2 Q#i BM M7 2Ci T QM /BM; K T
h` MbH i2 #v amHHBp Möb /B+iBQM `v, ? `K
;`QmTb- BM T `iB+mH ` /Bb+`2i2 ? `KQMB+
r HFb QM ;`QmTbX

h?2 bBKTH2bi + b2, ?vT2`#QHB+ ` iBQM H K
CmHB b2i- r2 Q#i BM M7 2Ci T CM /BM; K T
h` MbH i2 #v amHHBp Möb /B+iBQM `v, ? `K
;`QmTb- BM T `iB+mH `- /Bb+`2i2 ? `KQMB+
r HFb QM ;`QmTbX
q? i r2 ? p2 /QM2, #mBH/ /Bb+`2i2 ? `KQMB
` M/QK r HF QM b2H7@bBKBH ` ;` T? bbQ+
/vM KB+bX

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Definition. A finitely generated group $G = < S >$ is called *Gromov hyperbolic* if the Cayley graph satisfies the δ-*thin triangle property*, i.e.

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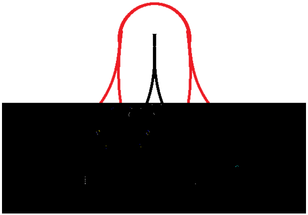
For each geodesic triangle, each edge is contained in the δ-neighborhood of other 2 edges.



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The *Gromov boundary* is

$$\partial X = \left\{ \{x_n\}_{n \in \mathbb{Z}_+} : \lim_{n,m \to \infty} \langle x_n, x_m \rangle_o = \infty \right\} / \sim,$$

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The *horofunction boundary* is the boundary of the embedding image of $y \mapsto \beta(\cdot, y)$ w.r.t. pointwise convergence topology.

Random walk and Martin boundary

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$Z_0, \cdots, Z_n, \cdots$ - random variables.

$$G(x, y) = \quad (\#\{n : Z_n = y\}) = \sum_{n=0}^{\infty} p^{(n)}(x, y) \text{ - the Green function.}$$

$F(x, y) = \quad (\exists n \geqslant 0, Z_n = y)$.

$$K(x, y) = \frac{G(x, y)}{G(o, y)} = \frac{F(x, y)}{F(o, y)} \text{ - the Martin kernel.}$$

Martin boundary $\partial_M X$ is constructed by

$$y_n \longrightarrow \xi \in \partial_M X \iff K(\cdot, y_n) \overset{\text{pointwise}}{\longrightarrow} K(\cdot, \xi).$$

[redacted]

For every positive harmonic function h on X, there is a positive Borel measure ν_h on $\partial_M X$ such that

$$h(x) = \int_{\partial_M X} K(x, \xi) \mathrm{d}\nu_h(\xi). \tag{2.1}$$

Relations between the boundaries

[redacted]

For uniformly irreducible, finite range random walks on hyperbolic graph, Martin boundary ≅ Gromov boundary.

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uniformly irreducible - $\exists c_0, v$, s.t., $d(x, y) \leqslant 1 \Rightarrow \sum_{j=0}^{v} p^{(j)}(x, y) \geqslant c_0$.

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Green metric - $d_G(x, y) := -\log F(x, y)$.

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The Martin boundary is exactly the horofunction boundary w.r.t. Green metric d_G, which is hyperbolic, and Q.I. to d.

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Conformal measure

Patterson-Sullivan measure

$$\mu_s = \lim_{n\to\infty} \left(\sum_{|g|\leqslant n} e^{-s|g|} \right)^{-1} \left(\sum_{|g|\leqslant n} e^{-s|g|} \delta_g \right).$$

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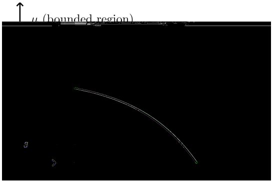

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Fact: μ is transient $\implies$ $\mu^{(n)}$ "converges" to a boundary distribution μ_h.

[redacted]

*If h is a **bounded harmonic** function on X, t .*

Conformal measure v.s. Harmonic measure

dilute a (weighted) uniform distribution $\Longleftrightarrow$ iterate a transition probability.

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$l_G = \lim_{n\to\infty} (d_G(1, Z_n(g)))/n, l = \lim_{n\to\infty} (d(1, Z_n(g)))/n$ - *drift*

[redacted]

The following are equivalent:

- *The equality of $l_G \leqslant vl$ holds.*
- $\mu_v \asymp \mu_h$.
- μ_h *is quasi-conformal.*

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Distance expanding dynamical systems

Definition. $f : X \to X$ is called λ-distance expanding if

$$\exists \xi > 0 \forall d(x,y) < \xi, d(fx, fy) \geqslant \lambda d(x,y).$$

$S = \{R_1, \cdots, R_n\}$ - Markov partition

- $\text{int } R_i \cap \text{int } R_j = \emptyset$ if $i \neq j$;
- $R_i = \overline{\text{int } R_i}$;
- $f(\text{int } R_i) \cap \text{int } R_j \neq \implies f(\text{int } R_i) \supset \text{int } R_j$.

$A_{R_i R_j} = 1 \iff f(\text{int } R_i) \supset \text{int } R_j$.
semi-conjugacy
$\left(\Sigma_A^+ = \{(u_n)_{n \geqslant 0} \in S^{\geqslant 0} : A_{u_i u_{i+1}} = 1\}, \sigma_A\right) \to (X, f)$.

Tile Graph

Vertices:
$\Gamma^0 = S^\omega = \{u_0 \cdots u_n := u_0 \cap \cdots \cap f^{-n} u_n : A_{u_i u_{i+1}} = 1\} \cup \{\emptyset\}$.
(called *tiles*)
Edges: $u - v$ if $\big||u| - |v|\big| \leqslant 1$ and $u \cap v \neq \emptyset$.
d is called a *visual metric* if for some $\Lambda > 1$,

- $\mathrm{dist}(x, y) \gtrsim \Lambda^{-n}$, where x, y are disjoint n-tiles (in the sense of closed subsets);
- $\mathrm{diam}(x) \asymp \Lambda^{-|x|}$;

Fact. Γ is Gromov hyperbolic with Gromov boundary X, and the visual metric is Hölder equivalent to d.

Tile graph

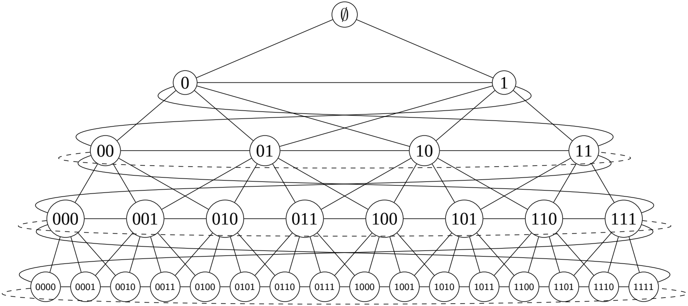

Figure: The tile graph of the doubling map on the circle.

i` MbBiBQM T: Q!#M #(H)B Bvb + HH2/ mQM?B 7pQ2`K H
#Q m M / 2 / b m9T*y-Q(`t;i v) > y =) /(t; v) 6 *yX

i` MbBiBQM TT: Q!#M #(H)BBvb + HH2/ mQM?B7pQ2`KH
#QmM/2/ b m9T*$_y$-Q(`t;iv) > y=) /(t; v) 6 *$_y$X
TBb + H @2B/M pB7B;6Q`t2 - (T(t)) = T(t)X
U: m_y m_M7! m_R m_MV

i` MbBiBQM TT: Q!#M #(H)B Bvb + HH2/ imQM?B7pQ2`KH
#QmM/2/ bm9T*$_y$-Q(`t;i v) > y =) /(t; v) 6 *$_y$ X
TBb + F7 l@2B/M pB 7B;76Q1`t2 - (T(t)) = T(t) X
U: m$_y$ m$_M$7! m$_R$ m$_M$V
q2 KQ`2Qp2` bbmK2 i? i,
h?2 ` M/QK r HF Hr vb ;Q2b b; v/2Q r-MB;T(7t; Q) > yHH
i?2jMy > jtj X
h?2 ` M/QK r HF BM?2`Bib i?2 2/;2 b; Q27 i?B27;`
jy = jtj + R M/(t; v) = R - i?2 T(Mv) > y X

[redacted]

h?2 J `iBM #QmM/ `v Q7 i?2 ` M/QK r HF QM
bm`D2+iBp2Hv iQ i?2 T? b2 bT +2,

[redacted]

amTTQb2 i?2 is7Bb M QT2M i` MbBiBp2 /BbiM
K TQM +QKT +i K2(is;B)+ bBTb+2bm{+B2MiHv }
J `FQp T `iBiB(Q;T)MB bM/M/QK r HF QM i?2 iBH2
=(7) rBi? T b iBb7vBM; i?2 bbmKTiBQMb
h?2M i?2`2 Bb M im` H `QmK`D?2+J B`QBM #QmM
@J Q(7;T) iQ sX

Ab i?2 J `iBM #QmM/ `v ?QK2QKQ`T?B+ iQ i?
LQi i`m2X q2 T`QpB/2 +QmMi2`2t^k K TTHQ2X 1p
i?2 mMBi +B`+H2- Bi Bb MQi i`m2 r?2 TMBi?2 i`
mM# H M+2/X h?2 ` /B H ;`Qri? ` i2 Q7 i?2 :`
bBM;H2t2 Q bBKMi #2 /Bz2`2MiX
>Qr2p2`- B7 i?2 ` M/QK r HF Bb B``2/m+B#H2
BKTHB2b i?2 ?QK2QKQ`T?BbK Q7 i?2 J `iBM
bT +2X

" b 2/ QM i?2 bvKTiQiB+ [m MiBiB2b BM+Hm/

$H_{:=} \lim_{M+1} M^{R} \log(:(w_y; w_M)) \quad M/h_{:=} \lim_{M+1} M^{R} \int w_M;$ UkXkV

r2 + M ;Bp2 7Q`KmH Q7 i?2 7` +i H /BK2Mb
K2 bm`2X

[redacted]

lM/2` i?2 MQi iBQMb M/ ?vTQi?2b2b #Q 2-
M @pBbm H7KQ2i`Bm{+B2MiHv bK> H-Hi?+2QMbi
i?2 T +FBM; /BK2MbBQM Q7 i?2 Q?M`sKQbM2B[-mKH

$$\dim_S = \frac{H_{:}}{H} X$$

h?2 T +FBM; /BK2MbBQM Q7 K2 bm`2 Bb ,

$$\dim_S = \inf f \dim_S( ) : \quad s; ( ) > y g = \inf f \dim_S( ) : \quad s; ( ) =$$

h?2 T +FBM; /BK2MbBQM Bb 2[m H iQ i?2 bm
HQ+ H /BK2MbBQMX

h?2 ? `KQMB+ KB2b@[m`2bB@BMp `B MiX A7 r2
bbmK2 i(?;v) > y BKTHB2b i?2 bB@BMp `B
*QM7Q`K H .vM KB+ H avbi2Kb 4 MmK#2` Q7 T`2BK ;2bV
[m bB@+QM7Q`K H KBiv
Zm2biBQMb,

*QM7Q`K H K2 bm`2fK2 bm`2 Q7 ?K tBK H ? `KQMB+ K2 bm`2\

Ab i?2 > mb/Q`z /BK2MbBQM Q7 i?2 ? `KQ

Ab i?2 7mM/ K2Mi H BM2[m HBiv biBHH i` mM/2`bi M/BM;\

* M r2 TTHv i?2 i?2`KQ/vM KB+ H 7Q`K H ? `KQMB+ K2 bm`2 b K2 bm`2 Q7 K tBK i?2 TQi2MiB H\