Weak universality results for singular SPDEs

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KPZ equation

The 1+1 dimensions KPZ equation is given by

$$\partial_t h = \partial_x^2 h + \mu(\partial_x h)^2 + \xi.$$

- $(t, x) \in \mathbf{R}^+ \times \mathbf{T}$.
- ξ is the space-time white noise.
- μ describes the strength of asymmetry.

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Renormalization: $\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + \mu(\partial_x h_\varepsilon)^2 + \xi_\varepsilon - C_\varepsilon$.

- $C_\varepsilon = \frac{c}{\varepsilon} + \mathcal{O}(1)$.

Theorem(Hairer 13' 14') h_ε converges in probability in some sense as $\varepsilon \to 0$ independent of regularization.

- The solution is defined as the limit, denote the solution family by KPZ(μ).

Weak universality for KPZ equation

Microscopic model: $\partial_t \widetilde{h} = \partial_x^2 \widetilde{h} + \sqrt{\varepsilon} F(\partial_x \widetilde{h}) + \widetilde{\xi}$.

- $\sqrt{\varepsilon}$: weak asymmetry.
- F: even function.
- $\widetilde{\xi}$: smooth space-time Gaussian field.

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Rescaled process $h_\varepsilon := \sqrt{\varepsilon}\widetilde{h}\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) - C_\varepsilon t$ solves

$$\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + \varepsilon^{-1} F(\sqrt{\varepsilon}\partial_x h_\varepsilon) + \xi_\varepsilon - C_\varepsilon.$$

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Theorem(Hairer-Quastel 18') If F is an even polynomial, then there exists $C_\varepsilon \to +\infty$ such that $h_\varepsilon \to KPZ(\mu)$.

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Theorem(Gubinelli-Perkowski 16', Yang 23'): If F is Lipschitz, then Hairer-Quastel universality holds in law at stationarity/non-stationarity.

- Require knowledge of invariant measure.
- The microscopic process should be Markovian.

Weak universality for KPZ equation

Heuristic explanation: Let Z_ε solves $\partial_t Z_\varepsilon = \partial_x^2 Z_\varepsilon + \xi_\varepsilon$ and $\Psi_\varepsilon = \partial_x Z_\varepsilon$. Then $u_\varepsilon := h_\varepsilon - Z_\varepsilon$ solves

$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + \varepsilon^{-1} F(\sqrt{\varepsilon}\Psi_\varepsilon + \sqrt{\varepsilon}\partial_x u_\varepsilon) - C_\varepsilon.$$

- $\sqrt{\varepsilon}\Psi_\varepsilon$ is asymptotically distributed as $\mathcal{N}(0, \sigma^2)$.
- $\partial_x u_\varepsilon$ has order $\varepsilon^{-\kappa}$.

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Taylor expanding $\varepsilon^{-1} F(\sqrt{\varepsilon}\Psi_\varepsilon + \sqrt{\varepsilon}\partial_x u_\varepsilon)$:

$$\underbrace{\varepsilon^{-1} F(\sqrt{\varepsilon}\Psi_\varepsilon) - C_\varepsilon}_{\mu \Psi^{\diamond 2}} + \underbrace{\varepsilon^{-\frac{1}{2}} F'(\sqrt{\varepsilon}\Psi_\varepsilon)}_{2\mu\Psi} \partial_x u_\varepsilon + \frac{1}{2} \underbrace{F''(\sqrt{\varepsilon}\Psi_\varepsilon)}_{2\mu} (\partial_x u_\varepsilon)^2 + \mathcal{O}(\varepsilon^{\kappa}).$$

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Question: Does the weak universality result of Hairer and Quastel hold for $F \in \mathcal{C}^{2+}$?

Arbitrary Nonlinearities

Theorem(Hairer-Xu 19'): For KPZ equation, if even function $F \in \mathcal{C}^{7+}$ with polynomial growth, then there exist $C_\varepsilon \to +\infty$ and μ such that $h_\varepsilon \to KPZ(\mu)$.

$$\begin{array}{ccc} \text{regularity structure} & + & \text{convergence of stochastic terms} \\ \Uparrow & & \Uparrow \\ F \in \mathcal{C}^{2+} & & F \in \mathcal{C}^{7+} \end{array}$$

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Theorem(Kong-Z. 22+) The same result holds for even function $F \in \mathcal{C}^{2+}$ with polynomial growth.

Weak universality for Φ^4_3 equation

Rescaled process ϕ_ε solves

$$\partial_t \phi_\varepsilon = \Delta \phi_\varepsilon - \varepsilon^{-\frac{3}{2}} V'(\sqrt{\varepsilon}\phi_\varepsilon) + \xi_\varepsilon + C_\varepsilon \phi_\varepsilon.$$

Take $u_\varepsilon := \phi_\varepsilon - \Psi_\varepsilon$ where Ψ_ε solves $\partial_t \Psi_\varepsilon = \Delta \Psi_\varepsilon + \xi_\varepsilon$. Then u_ε solves

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$$+ \underbrace{\frac{1}{6} V^{(4)}(\sqrt{\varepsilon}\Psi_\varepsilon)}_{\mu} u_\varepsilon^3 + \mathcal{O}(\varepsilon^\kappa).$$

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Question: How about arbitrary $V \in \mathcal{C}^{4+}$?

Arbitrary Nonlinearities

Theorem(Furlan-Gubinelli 19'): For Φ^4_3 equation, if even function $V \in \mathcal{C}^{10+}$ with exponential growth, then there exist $C_\varepsilon \to +\infty$ and μ such that $\phi_\varepsilon \to \Phi^4_3(\mu)$.

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Hairer-Xu 19': $\left|\mathbf{E}\prod_{j=1}^{2n}(\sin(\theta X_j))\right| \lesssim \theta^{2n}\mathbf{E}\prod_{j=1}^{2n} X_j = \mathbf{E}\prod_{j=1}^{2n}(\theta X_j)$.

Hairer-Xu 19':

$$\left|\mathbf{E}\prod_{j=1}^{2n}[\sin(\alpha X_j)\cdot \mathcal{T}_{(1)}(\cos(\beta Y_j))]\right| \lesssim (|\alpha|+|\beta|)^{6n}\mathbf{E}\prod_{j=1}^{2n}(X_j Y_j^{\diamond 2}). \quad (\star)$$

- Similar bounds hold for multi-frequency.
- We need $(\star)$ to prove the convergence of stochastic objects like

$$\varepsilon^{-\frac{3}{2}}\int K(x-y)F'(X)\mathcal{T}_{(1)}(F(Y))dy \qquad \text{in } \mathcal{C}^{-\frac{1}{2}-\kappa}$$

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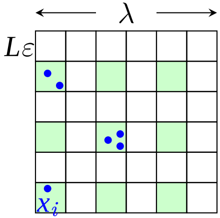
Xu 18': The bound with one frequency θ is independent of θ.
Proposition(Kong-Z. 22+) An integral version of $(\star)$ holds with bound independent of α and β.

Final Solution:

$$\varepsilon^{-\frac{3}{2}} \| \int \int K(x-y) \sin(\alpha X) \mathcal{T}_{(1)}(\cos(\beta Y)) \varphi^{\lambda}(x) \sum_{Q \in A} \mathbf{1}_{x \in Q} dx dy \|_{2n}$$

$$\triangleq \varepsilon^{-\frac{3}{2}} \| \sum_{Q \in A} Z_Q \|_{2n} = \varepsilon^{-\frac{3}{2}} \Big(\sum_{\sigma:[2n]\to A} \underbrace{\mathbf{E} Z_{\sigma(1)} \cdots Z_{\sigma(2n)}} \Big)^{\frac{1}{2n}}$$

$$= \int \int (\cdots) \mathbf{E} \prod_{i=1}^{2n} \Big(\sin(\alpha X_i) \mathcal{T}_{(1)}(\cos \beta Y_i) \mathbf{1}_{x_i \in \sigma(i)} \Big) d\vec{x} d\vec{y}$$



Whether there exists singleton depends only on σ.

Bad σ is rare.

$\mathcal{S} := \{\sigma : \exists Q, |\sigma^{-1}(Q)| = 1\}$

$|\mathcal{S}^c| \sim \left(\frac{\lambda}{\varepsilon}\right)^{nd}$

Solution

Furthermore, by Holder inequality we get

$$\sum_{\sigma \notin \mathcal{S}} \mathbf{E} Z_{\sigma(1)} \cdots Z_{\sigma(2n)} \leq \sum_{\sigma \notin \mathcal{S}} \|Z_{\sigma(1)}\|_{2n} \cdots \|Z_{\sigma(2n)}\|_{2n}.$$

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We have a uniform bound on $\|Z_Q\|_{2n}$.

$$\begin{aligned}
\|Z_Q\|_{2n} &= \| \int \int_{x \in Q} K(x-y) \sin(\alpha X) \mathcal{T}_{(1)}(\cos(\beta Y)) \varphi^\lambda(x) dx dy \|_{2n} \\
&\le \int_{x \in Q} \varphi^\lambda(x) \| \int K(x-y) \sin(\alpha X) \mathcal{T}_{(1)}(\cos(\beta Y)) dy \|_{2n} dx \\
&\lesssim \varepsilon^{-\kappa} \int_{x \in Q} \varphi^\lambda(x) \| \int K(x-y) \, Y^{\diamond 2} dy \|_{2n} dx \\
&\lesssim \varepsilon^{-\kappa} \left(\frac{\varepsilon}{\lambda}\right)^d \varepsilon
\end{aligned}$$

Solution

We would get

$$
\begin{aligned}
&\Big(\sum_{\sigma\in\mathcal{S}^c}\|Z_{\sigma(1)}\|_{2n}\cdots\|Z_{\sigma(2n)}\|_{2n}\Big)^{\frac{1}{2n}}\\
\le&|\mathcal{S}^c|^{\frac{1}{2n}}\sup_Q\|Z_Q\|_{2n}\\
\lesssim&\Big(\frac{\lambda}{\varepsilon}\Big)^{d/2}\Big(\frac{\varepsilon}{\lambda}\Big)^{d}\varepsilon^{1-\kappa}\\
=&\Big(\frac{\varepsilon}{\lambda}\Big)^{d/2}\varepsilon^{1-\kappa}\\
\lesssim&\varepsilon^{\frac{3}{2}-\kappa}\lambda^{-\frac{1}{2}}
\end{aligned}
$$

Further questions

- Weak universality results for non-Gaussian noise ξ.
  Hairer-Shen 17': If F is even polynomial, then h_ε converges to KPZ for non-Gaussian $\widetilde{\xi}$ (CLT).
  Question: How about general F with non-Gaussian $\widetilde{\xi}$?

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- Weak universality results for non-Gaussian noise ξ.
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- Weak universality results for $F(x) = |x|$ without the use of invariant measure.