#### Absolute purity in motivic homotopy theory

#### Fangzhou Jin joint work with F. Deglise, J. Fasel and A. Khan

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#### The absolute purity conjecture

Grothendieck's **absolute (cohomological) purity conjecture** (SGA5, Expose I 3.1.4) is the following statement: if  $i : Z \to X$  is a closed immersion between noetherian regular schemes of pure codimension  $c, n \in \mathcal{O}(X)$  and  $= \mathbb{Z} = n\mathbb{Z}$ , then the etale cohomology sheaf supported in Z with values in can be computed as

$$\mathcal{H}_{Z}^{q}(X_{\text{et}}; \cdot) = \begin{cases} i \quad z(-c) & \text{if } q = 2c \\ 0 & \text{else} \end{cases}$$

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In other words,  $i^{!} = z(-c)[-2c]$ . This conjecture has been solved by Gabber.

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Gabber(1986): general case (written by Fujiwara)
 Based on Thomason's method + rigidity for algebraic
 K-theory

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- Our work: study absolute purity in the framework of motivic homotopy theory.
- Main result: the absolute purity in motivic homotopy theory is satis ed with rational coe cients in mixed characteristic.

## Motivic homotopy theory

- The motivic homotopy theory or A<sup>1</sup>-homotopy theory is introduced by Morel and Voevodsky (1998) as a framework to study cohomology theories in algebraic geometry, by importing tools from algebraic topology
- Idea: use the a ne line A<sup>1</sup> as a substitute of the unit interval to get an algebraic version of the homotopy theory
- Can be used to study cohomology theories such as algebraic *K*-theory, Chow groups (motivic cohomology) and many others
- Advantage: has many a lot of structures coming from both topological and algebraic geometrical sides

## Aspects of applications in various domains

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- Non-commutative geometry and singularity categories (Tabuada, Blanc-Robalo-Toen-Vezzosi)

## Some topological background

• A **spectrum**  $\mathbb{E}$  is a sequence  $(E_n)_{n \ge \mathbb{N}}$  of pointed spaces (e.g. CW-complexes or simplicial sets) together with continuous maps  $_n : S^1 \land E_n \to E_{n+1}$  called **suspension maps** 

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- Examples: Suspension spectra  ${}^7 X$  for  $X \in Top$ , in particular sphere spectrum S; HA Eilenberg-Mac Lane spectrum for a ring A; MU complex cobordism spectrum
- From an ∞-categorical point of view, the category of spectra is the stabilization of the category of spaces, and is the universal stable (triangulated) category

### The unstable motivic homotopy category

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- Bigraded  $\mathbb{A}^1$ -homotopy sheaves: for  $X \in \mathbf{H}(S)$ ,  $\mathbb{A}^1_{a,b}(X)$  is the Nisnevich sheaf on  $Sm_S$  associated to the presheaf

$$U \mapsto [U \wedge S^{a \ b} \wedge \mathbb{G}_{m}^{b}; X]_{\mathbf{H}_{\bullet}(S)}$$

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- SH(S) is the universal stable ∞-category which satis es Nisnevich descent and A<sup>1</sup>-invariance (Robalo, Drew-Gallauer)

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#### Motivic spectra

Every object in **SH**(*S*) represents a bigraded cohomology theory  $\mathbb{E}^{p,q}(U) = [U_{\mathcal{F}}(S^{1})^{\wedge (p-q)} \wedge (\mathbb{G}_{m})^{\wedge q} \wedge \mathbb{E}]_{SH(S)}$ 

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- The 1-line is also computed (Rondigs-Spitzweck- stvaer):

$$0 \rightarrow K_{2 n}^{M} = 24 \rightarrow n+1, n(\mathbb{1}_{k}) \rightarrow n+1, nf_{0}(\mathbf{KQ})$$

## The six functors formalism

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 $f_!$ : **SH**(X)  $\rightleftharpoons$  **SH**(Y) :  $f^!$ 

There is also a pair  $(\otimes; \underline{Hom})$  of adjoint functors inducing a closed symmetric monoidal structure on **SH** 

# The six functors formalism

- Originates from Grothendieck's theory for *I*-adic sheaves (SGA4), and worked out in the motivic setting by Ayoub and Cisinski-Deglise
- For any morphism of schemes *f* : *X* → *Y*, there is a pair of adjoint functors

 $f : \mathbf{SH}(Y) \rightleftharpoons \mathbf{SH}(X) : f$ 

For any separated morphism of nite type  $f : X \to Y$ , there is an additional pair of adjoint functors

 $f_{!}: \mathbf{SH}(X) \rightleftharpoons \mathbf{SH}(Y): f^{!}$ 

There is also a pair  $(\otimes; \underline{Hom})$  of adjoint functors inducing a closed symmetric monoidal structure on **SH** 

• They satisfy formal properties axiomatizing important theorems such as duality, base change and localization.

### Thom spaces and relative purity

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- In the presence of an *orientation*, we recover the usual relative purity

#### Orientations

An absolute motivic spectrum is the data of E<sub>X</sub> ∈ SH(X) for every scheme X, together with natural isomorphisms
 *f* E<sub>X</sub> ≃ E<sub>Y</sub> for every morphism *f* : Y → X

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- A theory of *fundamental classes* aims at establishing a cohomological intersection theory
- For oriented spectra, Deglise de ned fundamental classes using Chern classes

## **Bivariant groups**

For f : X → S be a separated morphism of nite type,
 v ∈ K<sub>0</sub>(X) and E ∈ SH(S), de ne the E-bivariant groups (or Borel-Moore E-homology) as

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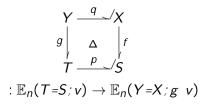
 $\mathbb{E}_n(X=S,v) = [f_!Th(v)[n],\mathbb{E}]_{\mathbf{SH}(S)}$ 

• If S is a eld and  $\mathbb{E} = \mathbf{H}\mathbb{Z}$ , then  $\mathbb{E}_i(X=S;v) = CH_r(X;i)$  are the higher Chow groups, where r is the virtual rank of v

Grothendieck's absolute purity conjecture Motivic homotopy theory The fundamental class

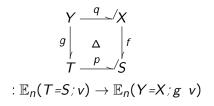
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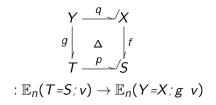


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• Product: if  $\mathbb{E}$  has a ring structure,  $X \xrightarrow{f} Y \xrightarrow{g} S$ 

$$\mathbb{E}_m(X=Y;w)\otimes\mathbb{E}_n(Y=S;v)\to\mathbb{E}_{m+n}(X=S;w+f~v)$$

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- The construction uses the deformation to the normal cone

## Euler class and excess intersection formula

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• Motivic Gauss-Bonnet formula (Levine, Deglise-J.-Khan) For  $p: X \rightarrow S$  a smooth and proper morphism

$$(X=S) = p \ e(T_p)$$

where (X=S) is the categorical Euler characteristic

## The absolute purity property

• We say that an absolute spectrum  $\mathbb{E}$  satis es **absolute purity** if for any closed immersion  $i : Z \to X$  between regular schemes, the purity transformation  $\mathbb{E}_Z \otimes \text{Th}(_f) \to f^! \mathbb{E}_X$  is an isomorphism

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 From this property Cisinski-Deglise deduce that the rational motivic Eilenberg-Mac Lane spectrum HQ also satis es absolute purity, mainly because HQ is a direct summand of KGLQ by the Grothendieck-Riemann-Roch theorem

### The Main result

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The \switching factors" endomorphism of P<sup>1</sup> ∧ P<sup>1</sup> induces a decomposition of the sphere spectrum 1<sub>Q</sub> into the direct sum of the plus-part 1<sub>+,Q</sub> and the minus-part 1<sub>,Q</sub> (Morel)

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- The +-part  $\mathbb{1}_{+,\mathbb{Q}}$  agrees with  $\textbf{H}\mathbb{Q}$  (Cisinski-Deglise)
- Therefore it su ces to show that the minus part satis es aboslute purity

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- Similar to the Chern character, the Borel character (de ned by Deglise-Fasel) induces a decomposition of KQ<sub>Q</sub>, where 1 ,<sub>Q</sub> can be identi ed as a direct summand
- This proves the absolute purity of  $\mathbb{1}_{\mathbb{Q}}$  when 2 is invertible on the base scheme, since KQ is only well-de ned in this case

### The second proof

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  SH(X) , vanishes by a theorem of Morel/Bachmann
- The key lemma then reduces the absolute purity of 1 ,Q in mixed characteristic to the case of Q-schemes, which can be proved using Popescu's theorem: a closed immersion of a ne regular schemes over a perfect eld is a limit of closed immersions of smooth schemes

- Our method can be used to deduce the following new results in mixed characteristic:
  - The six functors preserve constructible objects in the rational stable motivic homotopy category  $\textbf{SH}(\cdot;\mathbb{Q})$
  - The Grothendieck-Verdier duality holds for  $\mathbf{SH}(\cdot;\mathbb{Q})$
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- Related work: absolute purity of the sphere spectrum over a Dedekind domain (Frankland-Nguyen-Spitzweck, work in progress)

#### Thank you!

Fangzhou Jin joint work with F. Déglise, J. Fasel and A. Khan Absolute purity in motivic homotopy theory