

Projective Bundle Theorem in MW-Motives

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Motivation

Suppose $0 \leq i \leq n$, we have:

$$H^i(\mathbb{R}P^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or } i = n \text{ and } n \text{ is odd} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } i > 0 \text{ is even} \\ 0 & \text{else.} \end{cases}$$

Theorem (Fasel, 2013)

$$\widetilde{CH}^i(\mathbb{P}^n) = \begin{cases} GW(k) & \text{if } i = 0 \text{ or } i = n \text{ and } n \text{ is odd} \\ \mathbb{Z} & \text{if } i > 0 \text{ is even} \\ 2\mathbb{Z} & \text{else} \end{cases}$$

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Question

- A motivic explanation?
- How about projective bundles?

Chow Groups

- $CH^n(X) = \mathbb{Z}$ fcyycles of codimension n g/rational equivalence:

$$\bigoplus_{y \in X^{(n-1)}} k(y)^* \xrightarrow{\text{div}} \bigoplus_{y \in X^{(n)}} \mathbb{Z} \longrightarrow 0.$$

\vdots
 H
 $CH^n(X)$

- Projective bundle theorem:

$$CH^n(\mathbb{P}(E)) = \bigoplus_{i=0}^{rk(E)-1} CH^{n-i}(X) \quad \mathbb{P}(E) = \bigoplus_{i=0}^{rk(E)-1} \mathcal{O}(i)[2i].$$

- Chern class:

$$c_i(E) \in CH^i(X).$$

Chow-Witt Groups

Suppose X is smooth and $L \in \text{Pic}(X)$. We have the Gersten complex:

$$\begin{array}{ccccc} \bigoplus_{y \in X^{(n-1)}} \mathbf{K}_1^{MW}(k(y), L \otimes \mathcal{O}_y) & \xrightarrow{\text{div}} & \bigoplus_{y \in X^{(n)}} \mathbf{GW}(k(y), L \otimes \mathcal{O}_y) & \xrightarrow{\text{div}} & \bigoplus_{y \in X^{(n+1)}} \mathbf{W}(k(y), L \otimes \mathcal{O}_y) \\ & & \vdots & & \\ & & \widetilde{CH}^n(X, L) & & \end{array}$$

Chow-Witt Groups

- Suppose X is cellular. We have a Cartesian square:

$$\begin{array}{ccccc} \widetilde{CH}^n(X) & \longrightarrow & \ker(\partial) & & CH^n(X) & \longrightarrow & H^{2n}(X(\mathbb{C}), \mathbb{Z}) . \\ \downarrow & & \downarrow & & \searrow @ & & \\ H^n(X, \mathbb{I}^n) & \longrightarrow & CH^n(X)/2 & \longrightarrow & H^{n+1}(X, \mathbb{I}^{n+1}) & & \\ \downarrow & & & & & & \\ H^n(X(\mathbb{R}), \mathbb{Z}) & & & & & & \end{array}$$

- Pontryagin class
- Bockstein image of Stiefel-Whitney classes
- Orientation class

Four Motivic Theories

- Suppose $\mathbf{K} = MW, M, W, M/2$. We have a homotopy Cartesian:

$$\begin{array}{ccc} MW & \longrightarrow & W \\ \downarrow & & \downarrow \\ M & \longrightarrow & M/2 \end{array} .$$

Definition

Define the category of effective \mathbf{K} -motives over S with coefficients in R :

$$DM_{\mathbf{K}}^e = D[(X \quad A^1 \quad / \quad X)^{-1}]$$

where D is the derived category of Nisnevich sheaves with \mathbf{K} -transfers.

- $\mathbf{K} = MW \Rightarrow$ Milnor-Witt Motives
- $\mathbf{K} = M \Rightarrow$ Voevodsky's Motives

Four Motivic Theories

Theorem (BCDFØ, 2020)

For any $X \in \text{Sm}/S$ and $n \in \mathbb{N}$, we have

$$[X, Z(n)[2n]]_{\mathbf{K}} = \widetilde{CH}^n(X), CH^n(X), CH^n(X)/2$$

if $\mathbf{K} = MW, M, M/2$.

Theorem (Cancellation, BCDFØ, 2020)

Suppose $S = \text{pt}$. For any $A, B \in DM_{\mathbf{K}}^e$, we have

$$[A, B]_{\mathbf{K}} \stackrel{\otimes(1)}{\cong} [A(1), B(1)]_{\mathbf{K}}.$$

Basic Calculations

- $A^n = Z$.
- $G_m = Z \oplus Z(1)[1]$.
- $A^n \setminus \{0\} = Z \oplus Z(n)[2n - 1]$.
- $P^1 = Z \oplus Z(1)[2]$.
- $A^n / (A^n \setminus \{0\}) = P^n / (P^n \setminus \{pt\}) = Z(n)[2n]$.
- $E = X$ for any A^n -bundle E over X .

Hopf Map η

Definition

The multiplication map $G_m \times G_m \rightarrow G_m$ induces a morphism

$$G_m \times G_m \rightarrow G_m.$$

It's the suspension of a (unique) morphism $\eta \in [G_m, \mathbb{1}]$, which is called the Hopf map.

It's also equal, up to a suspension, to the morphism

$$\begin{array}{ccc} \mathbb{A}^2 \setminus \{0\} & \rightarrow & \mathbb{P}^1 \\ (x, y) & \mapsto & [x : y] \end{array}$$

Remark

The $\eta = 0$ if $K = M, M/2$, but never zero if $K = MW, W!$

$$\pi_3(S^2) = \mathbb{Z} \text{ Hopf}$$

MW-Motive of \mathbb{P}^n

Theorem (Y)

Suppose $n \geq 1$ and $p : \mathbb{P}^n \dashrightarrow \text{pt}$.

① If n is odd, there is an isomorphism

$$\mathbb{P}^n (p; c_n^{2i-1}; th_{n+1}) \cong R \bigoplus_{i=1}^{\frac{n-1}{2}} \text{cone}(\eta)(2i-1)[4i-2] \oplus R(n)[2n].$$

② If n is even, there is an isomorphism

$$\mathbb{P}^n (p; c_n^{2i}; th_{n+1}) \cong R \bigoplus_{i=1}^{\frac{n}{2}} \text{cone}(\eta)(2i-1)[4i-2].$$

Here $th_{n+1} = i_*(1)$ for some rational point $i : \text{pt} \dashrightarrow \mathbb{P}^n$.

$$c_n^{2i-1} : \mathbb{P}^n \rightarrow \text{cone}(\eta)(2i-1)[4i-2]$$

We have $\text{cone}(\eta) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$ in DM_M^e since $\eta = 0$. This implies

$$[\mathbb{P}^n, \text{cone}(\eta)(j)[2j]]_M = CH^j(\mathbb{P}^n) \oplus CH^{j+1}(\mathbb{P}^n).$$

We have an adjunction $\gamma^* : DM_{MW}^e \rightarrow DM_M^e : \gamma_*$.

Theorem (Y)

Suppose $j = 2i - 1 \leq n - 1$. The morphism

$$\gamma^* : \left[\mathbb{P}^n, \text{cone}(\eta)(j)[2j] \right]_{MW} \xrightarrow{c_n^j} \left[\mathbb{P}^n, \text{cone}(\eta)(j)[2j] \right]_M \xrightarrow{!} (c_1(\mathcal{O}(1))^k, c_1(\mathcal{O}(1))^{k+1})$$

is injective with $\text{coker}(\gamma^*) = \mathbb{Z}/2\mathbb{Z}$.

Splitness in MW-Motives

Definition

We say $X \in Sm/k$ splits in DM_{MW}^e if it's isomorphic to the form

Goal

Suppose E is a vector bundle. Find out the global definition of c_n^{2i-1} and th_{n+1} on $P(E)$.

Motivic Stable Homotopy Category $SH(k)$

- \mathbb{P}^1 spectra of simp. Nis. sheaves g /stable A^1 -equivalences.
- E -cohomologies:

$$[\Sigma^\infty X_+, E(q)[p]]_{SH(k)} = E^{p,q}(X).$$

- $H^n(X, \mathbf{K}_n) = H_{\mathbf{K}}^{2n;n}(X) = CH^n(X), \widetilde{CH}^n(X),$,
if $E = H Z, H\widetilde{Z},$.
- $(DM_{MW})_{\mathbb{Q}} = SH_{\mathbb{Q}}.$

Motivic Cohomology Spectra

Definition

Every motivic theory corresponds to a spectrum in $SH(k)$, namely

$$\begin{array}{cccc} H\tilde{Z} & HZ & H_W Z & HZ/2. \\ \downarrow & \downarrow & \downarrow & \downarrow \\ MW & M & W & M/2 \end{array}$$

The spectrum represents the $\text{cone}(\eta)$ (induces the same cohomologies) of, for example, MW-motive is denoted by $H\tilde{Z}/\eta$.

$$H\tilde{Z}/\eta$$

Theorem (Y)

We have a distinguished triangle

$$P^1 \wedge HZ \rightarrow H\tilde{Z}/\eta \rightarrow HZ \rightarrow HZ/2[2] \rightarrow P^1 \wedge HZ[1].$$

Remark

The triangle doesn't split since applying $\pi_2(\)_0$ we get an exact sequence of Nisnevich sheaves

$$0 \rightarrow Z/2Z \rightarrow O^* \rightarrow 2O^* \rightarrow 0.$$

$\eta_{MW}^i(X)$

Definition

$$\eta_{MW}^i(X) := [X, \text{cone}(\eta)(i)[2i]]_{MW} = [\Sigma^\infty X_+, H\tilde{Z}/\eta(i)[2i]]_{SH(k)}.$$

Theorem (Y)

If $R = \mathbb{Z}$ and ${}_2CH^{i+1}(X) = 0$, we have a natural isomorphism

$$\theta^i : CH^i(X) \rightarrow CH^{i+1}(X) \rightarrow \eta_{MW}^i(X).$$

Corollary

If $R = \mathbb{Z}[\frac{1}{2}]$, we have a natural isomorphism

$$\theta^i : CH^i(X)[\frac{1}{2}] \rightarrow CH^{i+1}(X)[\frac{1}{2}] \rightarrow \eta_{MW}^i(X)$$

for any $X \in \text{Sm}/k$.

a^k, b^k

Definition

Suppose $n \geq k + 1$ and k is odd. Define $a^k, b^k \in \mathbb{Z}$ by

$$(a^k c_1(O(1))^k, b^k c_1(O(1))^{k+1}) = \frac{1}{k!} [P^n, \text{cone}(\eta)(k)[2k]]_{MW} \cdot c_n^k.$$

They are independent of n .

$$c(E)^k : P(E) \quad ! \quad \text{cone}(\eta)(k)[2k]$$

Definition

Suppose E is a vector bundle of rank n over X , $R = \mathbb{Z}$, ${}_2CH^*(X) = 0$ and $k - n - 2$ is odd. Define $c(E)^k$ by

$$\frac{CH^k(P(E)) \quad CH^{k+1}(P(E))}{(a^k c_1(O(1))^k, b^k c_1(O(1))^{k+1})} \quad \begin{matrix} ! \\ 7! \end{matrix} \quad \frac{[P(E), \text{cone}(\eta)(k)[2k]]_{MW}}{c(E)^k} .$$

If $R = \mathbb{Z}[\frac{1}{2}]$, $c(E)^k$ is defined for all $X \in Sm/k$.

Projective Orientability

Recall SL^c -bundles are vector bundles E over X such that

$$\det(E) \in 2\text{Pic}(X).$$

Definition

Let E be an SL^c -bundle with even rank n over X . It's said to be projective orientable if there is an element $th(E) \in \widetilde{CH}^{n-1}(P(E))$ such that for any $x \in X$, there is a neighbourhood U of x such that $E|_U$ is trivial and

$$th(E)|_U = p^* th_n,$$

where $p : \mathbb{P}^{n-1} \rightarrow U \cong \mathbb{P}^{n-1}$.

Projective Orientability

- In Chow rings, we can always let $th(E) = c_1(O_{\mathbb{P}(E)}(1))^{n-1}$. But this doesn't work for Chow Witt rings!
- If E has a quotient line bundle, it's projective orientable.
- If E has a quotient bundle being projective orientable, it's projective orientable.
- Further characterization?

Projective Bundle Theorem

Theorem (Y)

Let E be a vector bundle of rank n over X . Suppose $2CH^*(X) = 0$ and X admits an open covering $\{U_i\}$ such that $CH^j(U_i) = 0$ for all $j > 0$ and i . Denote by $p: P(E) \rightarrow X$.

- ① If n is even and E is projective orientable, the morphism $(p, p^*c(E)^{2i-1}, p^*th(E))$

$$P(E) \rightarrow X \xrightarrow{\bigoplus_{i=1}^{\frac{n}{2}-1} X \text{ cone}(\eta)(2i-1)[4i-2]} X(n-1)[2n-2]$$

is an isomorphism.

- ② If n is odd, there is an isomorphism

$$P(E) \xrightarrow{(p, p^*c(E)^{2i-1})} X \xrightarrow{\bigoplus_{i=1}^{\frac{n-1}{2}} X \text{ cone}(\eta)(2i-1)[4i-2]} X(n-1)[2n-2].$$

Projective Bundle Theorem

Corollary

Let E is a vector bundle of odd rank n over X . If X is quasi-projective, we have

$$P(E) = X \oplus_{i=1}^{\frac{n-1}{2}} X \quad \text{cone}(\eta)(2i-1)[4i-2].$$

In particular, we have ($k = \min\{b^{\frac{i+1}{2}}, \frac{n-1}{2}\}$)

$$\widetilde{CH}^i(P(E)) = \widetilde{CH}^i(X) \oplus_{j=1}^k \widetilde{CH}^{i-2j+2}(X \times \mathbb{P}^2) / \widetilde{CH}^{i-2j+2}(X).$$

Projective Bundle Theorem

Theorem (Y)

Let E be a vector bundle of rank n over X . Suppose $2 \in R^\times$. Denote by $p : P(E) \rightarrow X$. If n is even and E is projective orientable, the morphism $(p_*, p^* c(E)^{2i-1}, p^* th(E))$

$$p_* : H^*(P(E)) \rightarrow H^*(X) \oplus_{i=1}^{\frac{n}{2}-1} H^*(X) \oplus cone(\eta)(2i-1)[4i-2] \oplus H^*(X)(n-1)[2n-2]$$

is an isomorphism.

In particular, we have $(k = \min\{b \frac{i+1}{2} c, \frac{n}{2} - 1\} g)$

$$\widetilde{CH}^i(P(E)) = \widetilde{CH}^i(X) \oplus_{j=1}^k \widetilde{CH}^{i-2j+2}(X \times P^2) / \widetilde{CH}^{i-2j+2}(X) \oplus \widetilde{CH}^{i-n+1}(X)$$

after inverting 2.

Blow-ups

Theorem (Y)

Suppose Z is smooth and closed in X , $n := \text{codim}_X(Z)$ is odd and Z is quasi-projective. We have

$$Bl_Z(X) = X \oplus_{i=1}^{\frac{n-1}{2}} Z \oplus \text{cone}(\eta)(2i-1)[4i-2].$$

In particular, we have $(k = \min\{b, \frac{n-1}{2}\}, \frac{n-1}{2}g)$

$$\widetilde{CH}^i(Bl_Z(X)) = \widetilde{CH}^i(X) \oplus_{j=1}^k \widetilde{CH}^{i-2j+2}(Z \times \mathbb{P}^2) \oplus \widetilde{CH}^{i-2j+2}(Z).$$

Thank you!