2020 6 13

1.
2.
3. motive

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2.
3. motive

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k: , S: k , S **广群** (groupoid) :

- S G,
- $b, s : G \to S$,
- $S \times S$ $\circ : G \times_{{}_s S^b} G \to G$.

k T,

$$(S(T) = \mathrm{Hom}(T, S), G(T) = \mathrm{Hom}(T, S), s, b, \circ)$$

, $S(T)$, $G(T)$.

: $S \times S$ $\varepsilon : S \to G$ "-1" $: G \to G$, .

.

S G **表示** $V \in \mathsf{Qcoh}(S)$, G .
G $u : s^*V \xrightarrow{\sim} b^*V$:

- $\varepsilon^* u$,
- u $\circ : G \times_{s\,S\,b} G \to G$ $\mathrm{pr}_1^* u \circ \mathrm{pr}_2^* u$.

S T, V ,
$(S(T), G(T), s, b, \circ)$:
$t : T \to S$ t^*V,
$g : G \to S$ $g^*u : s(g)^*V \to b(g)^*V$.

.

S G **表示** $V \in \mathrm{Qcoh}(S)$, G .

gerbe

S , Sch/k ,
(prestack). fpqc $\mathcal{G}_{S:G}$.

Sch/k $\mathcal{F}$,

Sch/k $\mathcal{F} \to \mathsf{Qcoh}/k$.

: k T, $\mathcal{F}(T) \to \mathsf{Qcoh}(T)$.

, $\mathsf{Rep}(\mathcal{G}_{S:G}) \simeq \mathsf{Rep}(S:G)$.

gerbe

S , Sch/k ,
(prestack). fpqc $\mathcal{G}_{S:G}$.

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, $\mathsf{Rep}(\mathcal{G}_{S:G}) \simeq \mathsf{Rep}(S:G)$.

gerbe ,
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$\mathcal{G}_{S:G}$ gerbe
$(s, b) : G \to S \times S$ fpqc .
G .
G S .

Deligne 1990 .

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k , k-**张量范畴** (tensor category)

k- $\mathcal{C}$ $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$:

1. $(\mathcal{C}, \otimes)$ $\mathbb{1}$.
2. $X \in \mathcal{C}$, $X^\vee$ $\delta : \mathbb{1} \to X^\vee \otimes X$ $\mathrm{ev} : X \otimes X^\vee \to \mathbb{1}$:
   $X \xrightarrow{\mathrm{id}\otimes\delta} X \quad X^\vee \quad X \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} X$ 和 $X^\vee \xrightarrow{\delta\otimes\mathrm{id}} X^\vee \quad X \quad X^\vee \xrightarrow{\mathrm{id}\otimes\mathrm{ev}} X^\vee$
   .
3. $\mathcal{C}$.
4. $k \xrightarrow{\sim} \mathrm{Hom}(\mathbb{1}, \mathbb{1})$.

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- (1) $\mathcal{T}, \mathcal{T}'$, $F : \mathcal{T} \to \mathcal{T}'$ **张量函子** ($\otimes$-functor)
  - $F(X) \otimes F(Y) \to F(X \otimes Y)$,
  - , .
- $\mathcal{T}$ **纤维函子** (fiber functor) $F : \mathcal{T} \to \mathrm{Qcoh}(S)$ S k- .
- $\mathcal{T}$ **淡中范畴** (tannakian category) $\omega : \mathcal{T} \to \mathrm{Qcoh}(S)$ $S \neq \emptyset$.

$u : T \to S$ k- , $\omega : \mathcal{T} \to \mathrm{Qcoh}(S)$.
$u^*\omega : \mathcal{T} \to \mathrm{Qcoh}(T)$.
$\mathcal{T}$
$\omega : \mathcal{T} \to \mathrm{Vect}(K)$, K k .

. (Deligne 1990)

$\mathcal{T}$: k , $\omega : \mathcal{T} \to \mathsf{Qcoh}(S)$: , $S/k \neq \emptyset$.

- $\underline{\mathrm{Aut}}_k^{\otimes}(\omega)$ $S \times S$.
- ω $\mathcal{T} \xrightarrow{\sim} \mathsf{Rep}(S : \underline{\mathrm{Aut}}_k^{\otimes}(\omega))$.
- , G $S \neq \emptyset$,
  $S \times S$.
  ω $\mathsf{Rep}(S : G)$ G ,
  ω : $G \xrightarrow{\sim} \underline{\mathrm{Aut}}_k^{\otimes}(\omega)$.

. (Barr-Back)

(L, R) , $LR \to \mathrm{id}, \mathrm{id} \to RL$,
$F = LR$, $LA \to LRLA = F(LA)$ LA F .
$L : \mathcal{A} \to \mathcal{B}$ $(f, g) : A \rightrightarrows A'$
(Lf, Lg) , $K = \ker(f, g)$ $LK = \ker(Lf, Lg)$.
L $\mathcal{A}$ F $\mathcal{B}$.

$_AM_B$ (A, B) , A , B
, B $_BM_A^\vee$ (B, A) .
$L : E \to E \otimes_A {}_AM_B$ A B ,
$R : F \to F \otimes_B {}_BM_A^\vee$, Barr-Back ,
A $_BM_A^\vee \otimes_A {}_AM_B$ B .

( )

k: , B_1, B_2: k . $\omega_i : \mathcal{C} \to (B_i)_{\mathrm{ptf}}$,

$(B_i)_{\mathrm{ptf}}$ B_i .

:(B_1, B_2) $L_k(\omega_1, \omega_2)$:

$X \in \mathcal{C}$, $\omega_1(X)^\vee \otimes \omega_2(X) \to L_k(\omega_1, \omega_2)$

$X, Y \in \mathcal{C}$, .

: .

$$L_k(\omega_1, \omega_3) \to L_k(\omega_1, \omega_2) \otimes L_k(\omega_2, \omega_3).$$

.

$L_k(\omega) = L_k(\omega, \omega)$, . .

, $\omega(X)^\vee \otimes \omega(X) \to L_k(\omega)$ $\omega(X) \to \omega(X) \otimes L_k(\omega)$

$L_k(\omega)$ $\omega(X)$.

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k , $\mathcal{A}$ k- *Abel* ,
, Hom .
k B, $\omega : \mathcal{A} \to (B)_{\mathrm{ptf}}$,

( )

( ).

$Y \in \langle X \rangle$ $\langle Y \rangle \subseteq \langle X \rangle$.
, $L_k(\omega|\langle Y \rangle) \to L_k(\omega|\langle X \rangle)$,
B .
$\mathcal{A}$ $\langle X \rangle$, $L_k(\omega)$
$L_k(\omega|\langle X \rangle)$ . . □

$\mathcal{A}$, ω ,
$L_k(\omega_1, \omega_2) \otimes L_k(\omega_1, \omega_2) \to L_k(\omega_1, \omega_2)$.
$\mathcal{A}$, $L_k(\omega_1, \omega_2)$
B . $\operatorname{Spec} L_k(\omega_1, \omega_2)$ $\underline{\mathrm{Hom}}_S^{\otimes}(\omega_1, \omega_2)$,
$S = \operatorname{Spec} B$.

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$\mathcal{T}$ $T_i : \mathcal{C} \to \mathcal{T}$,

$\Lambda_{\mathcal{T}}(T_1, T_2)$ $T_1(X)^\vee \otimes T_2(X)$ (coend), $\mathrm{Ind}(\mathcal{T})$

. , $\Lambda_k(T_1, T_2) = \Lambda_{\mathcal{T}\otimes\mathcal{T}}(T_1 \times \mathbb{1}, \mathbb{1} \times T_2)$.

$T : \mathcal{T} \otimes \mathcal{T} \to \mathcal{T}$, $T(\Lambda_k(\mathrm{id}, \mathrm{id})) = \Lambda_{\mathcal{T}}(\mathrm{id}, \mathrm{id})$,

$\omega(\Lambda_k(\mathrm{id}, \mathrm{id})) = L_k(\omega)$.

$\mathrm{ev} : X^\vee \otimes X \to \mathbb{1}$ $\Lambda_{\mathcal{T}}(\mathrm{id}, \mathrm{id}) \to \mathbb{1}$,

$\mathbb{1} \to \Lambda_{\mathcal{T}}(\mathrm{id}, \mathrm{id})$.

$\mathcal{T}$ $S = \operatorname{Spec} B$,

$\in \mathrm{Ind}(\mathcal{T})$, $\omega(X)$ S .

Grassmannian

$G_{\mathcal{O}}$, G_F Ind- .

$Gr_G = G_F/G_{\mathcal{O}}$. Ind- .

, G GL_n , Gr_G Gr_{GL_n} .

GL_n, Gr_{GL_n}

Grassmannian .

Gr_G : Beauville-Laszlo

X , $x \in X$, $X^* = X \backslash \{x\}$.

R, $X_R = X \otimes_k R$.

$$Gr_G(R) = \left\{ (\mathcal{E}, \beta) \begin{array}{l} \mathcal{E} \quad X_R \quad G\text{-} \quad \text{(torsor)} \\ \beta : \mathcal{E}|_{X_R^*} \to \mathcal{E}^\circ|_{X_R^*} \end{array} \right\} , \quad \text{isom}$$

$G_{\mathcal{O}}$

$\mathrm{ev} : G_{\mathcal{O}} \to G$ t 0 .
(B, T) Borel , $I = \mathrm{ev}^{-1}B$ G_F Iwahori ,
$G_{\mathcal{O}}$ Parahori . Cartan :
$G_{\mathcal{O}} \backslash G_F / G_{\mathcal{O}} = W \backslash (X_\bullet \rtimes W)/W = X_\bullet^+$.

$\lambda \in X_\bullet$, $\mathbb{G}_m \to T$ $\mathbb{G}_{m,F} \to T_F$.
t^λ $t \in \mathbb{G}_{m,F}$.

$G_{\mathcal{O}}$ Gr_G , $X_\bullet^+$.
$\lambda \in X_\bullet^+$, $G_{\mathcal{O}} t^\lambda G_{\mathcal{O}} / G_{\mathcal{O}}$ $G_{\mathcal{O}}$.
$Gr_G^\lambda = G_{\mathcal{O}} t^\lambda G_{\mathcal{O}} / G_{\mathcal{O}}$,
$\mathrm{Perv}_{G_{\mathcal{O}}}(Gr_G, A)$ $IC_{\overline{Gr_G^\lambda}}$.

$$Gr_G \times Gr_G \xleftarrow{p} G_F \times Gr_G \xrightarrow{q} G_F \times^{G_{\mathcal{O}}} Gr_G = Gr_G \tilde{\times} Gr_G \xrightarrow{m} Gr_G.$$

$\mathcal{F}, \mathcal{G} \in \mathsf{Perv}_{G_{\mathcal{O}}}(Gr_G, A)$, $Gr_G \tilde{\times} Gr_G$

$\mathcal{F} \tilde{\boxtimes} \mathcal{G}$ $p^*(\mathcal{F} \boxtimes \mathcal{G}) = q^*(\mathcal{F} \tilde{\boxtimes} \mathcal{G})$.

$\mathcal{F} \star \mathcal{G} = m_*(\mathcal{F} \tilde{\boxtimes} \mathcal{G})$, $\mathcal{F}$ $\mathcal{G}$.

$$Gr_G \times Gr_G \xleftarrow{p} G_F \times Gr_G \xrightarrow{q} G_F \times^{G_\mathcal{O}} Gr_G = Gr_G \tilde{\times} Gr_G \xrightarrow{m} Gr_G.$$

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$\mathcal{F} \star \mathcal{G} = m_*(\mathcal{F} \tilde{\boxtimes} \mathcal{G})$, $\mathcal{F}$ $\mathcal{G}$.

$\mathsf{Perv}_{G_{\mathcal{O}}}(Gr_G, A)$.

$\lambda \in X_\bullet^+, \quad IC_\lambda = IC_{\overline{Gr_G^\lambda}}.$

$\mathrm{Hom}(IC_\lambda, IC_\mu[1]) = 0 \qquad \lambda, \mu \in X_\bullet^+.$

:

.

- $\overline{Gr_G^\lambda} = \sqcup_{\nu \le \lambda} Gr_G^\nu, \quad \dim \overline{Gr_G^\lambda} = \dim Gr_G^\lambda = \langle 2\rho, \lambda \rangle,$
  2ρ .
- $\mathcal{H}^i(IC_\lambda) = 0 \qquad i \qquad \langle 2\rho, \lambda \rangle$.

( )

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- : G_F Kac-Moody , $G_{\mathcal{O}}$ Parahori , .
- , : t^{λ} . : Gr_G^{λ} , .
- , : α $\lambda, \lambda - \alpha^{\vee} \in X_{\bullet}^{+}$, $C_{\lambda,\alpha^{\vee}}$, $C_{\lambda,\alpha} \backslash \{t^{\lambda-\alpha^{\vee}}\} \subset Gr_G^{\lambda}$. : SL_2 , .

□

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Bruhat , $Gr_G = \sqcup_{w \in \widetilde{W}/W} IwG_{\mathcal{O}}/G_{\mathcal{O}}$.

$w_\lambda \in Wt^\lambda W$ $Iw_\lambda G_{\mathcal{O}}/G_{\mathcal{O}}$ Gr_G^λ .
$IC_\lambda = IC_{Iw_\lambda G_{\mathcal{O}}/G_{\mathcal{O}}}$.

$Fl_G = G_F/I \to Gr_G$, G/B^- .
$Iw_\lambda G_{\mathcal{O}}/G_{\mathcal{O}}$ $Iw_\lambda w_\circ I/I$.

, $IC_{IwI/I}$, $\overline{IwI/I}$
Bott-Samelson , .

. □

$\mathcal{F}, \mathcal{G} \in \mathsf{Perv}_{G_{\mathcal{O}}}(1[00011001]/2110001[()]/10[()))]/11202$

$U = [B, B]$ B , $\lambda \in X_\bullet$,
$S_\lambda = U_F t^\lambda G_\mathcal{O}/G_\mathcal{O}$ U_F Gr_G .

:

.

- $\overline{S_\lambda} = \sqcup_{\nu \leq \lambda} S_\nu$;
- $2\rho^\vee$, $\mathbb{G}_m \to T \hookrightarrow G$ $\mathbb{G}_m$ Gr_G .

$$S_\lambda = \left\{ x \in Gr_G \ \lim_{s \to 0} 2\rho^\vee(s) \cdot x = t^\lambda \right\}.$$

U_F^-

$$T_\lambda = \left\{ x \in Gr_G \ \lim_{s \to \infty} 2\rho^\vee(s) \cdot x = t^\lambda \right\}.$$

( )-Braden's hyperbolic localization theorem

.

$\mathbb{G}_m$ X , $\rho : \mathbb{G}_m \times X \to X$,
$S \in \mathsf{D}(X)$,
$\rho^* S = L \boxtimes S$, L $\mathbb{G}_m$.

. (Braden 2003)

X . $F = X^{\mathbb{G}_m}$,

$$X^+ = \left\{ x \in X \;\middle|\; \lim_{s \to 0} s \cdot x \textit{ exists} \right\}, \quad X^- = \left\{ x \in X \;\middle|\; \lim_{s \to \infty} s \cdot x \textit{ exists} \right\}.$$

$f^{\pm} : F \to X^{\pm}, g^{\pm} : X^{\pm} \to X, \pi^{\pm} : X^{\pm} \to F.$
$S \in \mathsf{D}(X)$ $\mathbb{G}_m$- , :

- $(f^{\pm})^* S = (\pi^{\pm})_* S, (f^{\pm})^! S = (\pi^{\pm})_! S,$
- $(f^-)^*(g^-)^! S \to (f^+)^!(g^+)^* S.$

( )

$\overline{Gr_G^\lambda}$,

$$H^k_{T_\mu}(Gr_G, IC_\lambda) \xrightarrow{\sim} H^k_c(S_\mu, IC_\lambda).$$

.

$k \neq \langle 2\rho, \mu \rangle$.

.

$$\lambda \in X_\bullet^+, \mu \in X_\bullet$$

$\dim \overline{Gr_G^\lambda} \cap S_\mu = \langle \rho, \lambda + \mu \rangle.$

$\mathcal{F} \in \mathsf{Perv}_{G_\mathcal{O}}(Gr_G)$

$H^k_c(S_\mu, \mathcal{F}) = 0, k > \langle 2\rho, \mu \rangle.$

$H^k_{T_\mu}(Gr_G, \mathcal{F}) = 0, k < \langle 2\rho, \mu \rangle.$

. □

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: $\mathcal{F} \in \mathsf{Perv}_{G_\mathcal{O}}(Gr_G)$

- $H_c^{\langle 2\rho,\mu\rangle}(S_\mu, \mathcal{F}) = H_c^{\langle 2\rho,\mu\rangle}(\overline{S_\mu}, \mathcal{F})$,
- $H_c^{\langle 2\rho,\nu\rangle}(\overline{S_\mu}, \mathcal{F}) = H_c^{\langle 2\rho,\nu\rangle}(\overline{S_\nu}, \mathcal{F}), \nu \leq \mu$,
- $H_c^{\langle 2\rho,\mu\rangle + \mathsf{odd}}(\overline{S_\mu}, \mathcal{F}) = 0$.

T_μ .

, $F_\mu = H_c^{\langle 2\rho,\mu\rangle}(S_\mu, -)$, $H(Gr_G, \mathcal{F}) = \bigoplus_{\mu \in X_\bullet} F_\mu(\mathcal{F})$.

, :

$H_{T_\mu}^{\langle 2\rho,\mu\rangle}(Gr_G, \mathcal{F})$ $H^{\langle 2\rho,\mu\rangle}(Gr_G, \mathcal{F})$ $H^{\langle 2\rho,\mu\rangle}$

.

Grassmannian

X , $Gr_{G,X}$

( ) ,

$$Gr_{G,X}(R) = \quad (x, \mathcal{E}, \beta) \begin{array}{l} x \in X(R), \mathcal{E} \quad X_R \qquad G\text{-} \\ \beta : \mathcal{E}|_{X_R \setminus \Gamma_x} \to \mathcal{E}^{\circ}|_{X_R \setminus \Gamma_x} \end{array} \qquad \text{isom}$$

Gr_{G,X^2}, Beilinson-Drinfeld Grassmannian.

:

- $Gr_{G,X^2}|_{X^2 \setminus \Delta} = (Gr_{G,X} \times Gr_{G,X})|_{X^2 \setminus \Delta}$.
- $Gr_{G,X^2}|_{\Delta} \simeq Gr_{G,X}$.

( )

$$Gr_{G,X}\tilde{\times}Gr_{G,X} \xrightarrow{m} Gr_{G,X^2}.$$

$U = X^2\backslash\Delta$, Δ .

$X = \mathbb{A}^1$, , $Gr_{G,X} = Gr_G \times X$,

$\mathcal{F} \in \mathsf{Perv}_{G_\mathcal{O}}(Gr_G)$ $\mathcal{F}_X \in \mathsf{Perv}_{G_{\mathcal{O},X}}(Gr_{G,X})$.

, $\mathcal{F}_X \star_X \mathcal{G}_X$.

, Δ $\mathcal{F} \star \mathcal{G}[2]$,

U $\mathcal{F} \boxtimes \mathcal{G}[2]$.

Gr_{G,X^2} .

$\mathbb{G}_m$ $Gr_{G,X}\tilde{\times}Gr_{G,X}$, $(t^{\mu_1}, t^{\mu_2})_{X^2}$.

m , Δ ,

$\mu \in X_\bullet$, $S_\mu(X^2)$.

( )

$S_\mu(X^2)$ Δ $\bigsqcup$ S_μ,
U $\bigsqcup_{\mu_1+\mu_2=\mu} S_{\mu_1} \times S_{\mu_2}$.

$\mathcal{F}_X \star_X \mathcal{G}_X$ $\mathcal{H}_c^{\langle 2\rho,\mu\rangle-2}(S_\mu(X^2)/X^2, \mathcal{F}_X \star_X \mathcal{G}_X)$.

Δ $H_c^{\langle 2\rho,\mu\rangle}(S_\mu, \mathcal{F}\star\mathcal{G}) = F_\mu(\mathcal{F}\star\mathcal{G})$.

U

$$H^{\langle 2\rho,\mu\rangle}\Big(\bigsqcup_{\mu_1+\mu_2=\mu} S_{\mu_1} \times S_{\mu_2}, \mathcal{F}\boxtimes\mathcal{G}\Big) = \bigoplus_{\mu_1+\mu_2=\mu} F_{\mu_1}(\mathcal{F}) \otimes F_{\mu_2}(\mathcal{G}).$$

( )

$S_\mu(X^2)$ Δ F S_μ,
U $\underset{\mu_1+\mu_2=\mu}{\mathrm{F}} S_{\mu_1} \times S_{\mu_2}$.

$\mathcal{F}_X \star_X \mathcal{G}_X$ $\mathcal{H}_c^{\langle 2\rho,\mu\rangle-2}(S_\mu(X^2)/X^2, \mathcal{F}_X \star_X \mathcal{G}_X)$.

Δ $H_c^{\langle 2\rho,\mu\rangle}(S_\mu, \mathcal{F} \star \mathcal{G}) = F_\mu(\mathcal{F} \star \mathcal{G})$.

U

$$H^{\langle 2\rho,\mu\rangle} \underset{\mu_1+\mu_2=\mu}{\mathrm{F}} S_{\mu_1} \times S_{\mu_2}, \mathcal{F} \boxtimes \mathcal{G} = \underset{\mu_1+\mu_2=\mu}{\mathrm{L}} F_{\mu_1}(\mathcal{F}) \otimes F_{\mu_2}(\mathcal{G}).$$

. :

- X^2 .
  (,) .
- $\mathcal{H}^k(Gr_{G,X^2}, \mathcal{F}_X \star_X \mathcal{G}_X)$.
  .

$\mathsf{Sat}_G = \mathsf{Perv}_{G_{\mathcal{O}}}(Gr_G)$.

$F = H(Gr_G, -)$ Sat_G

$\mathsf{SVect}(A)$,

$\mathsf{Vect}(A)$.

Sat_G $\mathsf{Rep}(\check{G}, A)$.

Sat_G , $\check{G}$.

Sat_G X, $\langle X \rangle$,

G .

Sat_G $\check{G}$.

$\check{G}$ $\check{\mathfrak{g}}$.

( )

T $\mathbb{G}_m^n$,

Gr_T $X_\bullet(T)$.

Sat_T $X_\bullet(T)$.

$F = P$.

$F = \bigoplus_{\mu\in X_\bullet} F_\mu$:

$\mathsf{Sat}_G \to \mathsf{Vect}_{X_\bullet}(A) \to \mathsf{Vect}(A)$. $P \to G$.

G $X_\bullet^+$, G .

( )

$2\rho \in X^\bullet = X^{\mathfrak{f}}_\bullet$, ,
$\mathfrak{G}$, Borel , $\mathfrak{b}$.

$\lambda \in X^+_\bullet$, IC_λ λ $F(IC_\lambda)$.
, $\mathfrak{G}$ λ , $X^+_\bullet$.

$X^+_\bullet = X^{\mathfrak{g}+}_\bullet$.

, $F(IC_\lambda)_\mu$ $\mu \leq \lambda$. $X^{\mathfrak{f}\bullet}$
(root lattice) $Q^{\mathfrak{f}}_+$ $Q^\vee_+$.

$\mathfrak{G}$.

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3 motive

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- Zeta

Hodge

k: 0 , $\overline{k}$, Galois $\Gamma = \mathrm{Gal}(\overline{k}/k)$.

X: k . ℓ, $H_\ell(X)$: X ℓ- ,

$H_{\mathrm{dR}}(X)$: X de Rham . $H_{\mathbb{A}}(X) = H_{\mathrm{dR}}(X) \times \underset{\ell}{\mathrm{Q}} H_\ell(X)$.

$\sigma : k \hookrightarrow \mathbb{C}$, $H_\sigma(X)$ $\sigma X = X \otimes_{k,\sigma} \mathbb{C}$ Betti .

: $H_\sigma(X) \otimes (\mathbb{C} \times \mathbb{A}_f) \xrightarrow{\sim} H_{\mathbb{A}}(\sigma X)$.

σ^* $H_\sigma(X) \hookrightarrow H_{\mathbb{A}}(\sigma X)$.

Hodge

k: 0 , $\bar{k}$, Galois $\Gamma = \mathrm{Gal}(\bar{k}/k)$.
X: k . ℓ, $H_\ell(X)$: X ℓ- ,
$H_{\mathrm{dR}}(X)$: X de Rham . $H_{\mathbb{A}}(X) = H_{\mathrm{dR}}(X) \times \prod_{\ell} H_\ell(X)$.

$\sigma : k \hookrightarrow \mathbb{C}$, $H_\sigma(X)$ $\sigma X = X \otimes_{k,\sigma} \mathbb{C}$ Betti .

: $H_\sigma(X) \otimes (\mathbb{C} \times \mathbb{A}_f) \xrightarrow{\sim} H_{\mathbb{A}}(\sigma X)$.

σ^* $H_\sigma(X) \hookrightarrow H_{\mathbb{A}}(\sigma X)$.

.

$H_{\mathbb{A}}^{2p}(X)(p)$ t **相对于 σ 的 Hodge 类**

$$\sigma^* t \in H^{2p}(\sigma X, \mathbb{Q})(p) \cap H^{p,p}(\sigma X).$$

$H_{\mathbb{A}}^{2p}(X)(p)$ t **绝对 Hodge 类**
$\sigma : k \hookrightarrow \mathbb{C}$ σ Hodge .

$C^p_{\mathrm{AH}}()$

V_k k ( ) .
CV_k : $h(X), X \in \mathsf{V}_k$,

$$\mathrm{Hom}(h(X), h(Y)) = \mathrm{Mor}^0(X, Y).$$

$\mathbb{Q}$- ,

$$h(X \sqcup Y) = h(X) \oplus h(Y), h(X \times Y) = h(X) \otimes h(Y).$$

, $\mathbb{1} = h(\mathsf{pt})$.

(effective)motive $\dot{\mathsf{M}}_k^+$
CV_k ( ).

( )

$\mathbb{P}^1$, $h(\mathbb{P}^1) = h^0(\mathbb{P}^1) + h^2(\mathbb{P}^1) = h(\mathrm{pt}) + L$.
$H(L) = \mathbb{Q}(-1)$,

$$\mathrm{Hom}(M, N) \xrightarrow{\sim} \mathrm{Hom}(M \otimes L, N \otimes L)$$

motive M, N.

L :

.

motive $\dot{\mathsf{M}}_k$:

1. $(M, m), M \in \dot{\mathsf{M}}_k^+, m \in \mathbb{Z}$;
2. $$\mathrm{Hom}_{\dot{\mathsf{M}}_k}((M, m), (N, n)) = \mathrm{Hom}_{\dot{\mathsf{M}}_k^+}(M \otimes L^{r-m}, M \otimes L^{r-n})$$
   $r \geq m, n$;
3. $\dot{\mathsf{M}}_k^+$.

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$\dot{\mathsf{M}}_k$.

.

$H^r(X)$ Hodge

$\mathrm{Mor}^0(X,X)$.

$$\mathrm{Hom}_{\dot{\mathsf{M}}_k^+}((h(x),p),(h(x),p)) = p\mathrm{Mor}^0(X,X)p$$

p .

Jannsen 1992 Lemma 2 . □

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$\dot{\mathsf{M}}_k$.

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, (rigid) .

L , .

$h(X), X \in \mathsf{V}_k$, X n,

$$h(X)^\vee = h(X)(n),$$

δ $[\Delta_X]$, ev .

$$p \in \mathrm{Hom}(h(X), h(X)),$$

${}^{\mathrm{t}}p$ $h(X)^\vee$.

$$(h(X), p)^\vee = (h(X)^\vee, {}^{\mathrm{t}}p).$$

□

$\dot{\mathsf{M}}_k$. $X \in \mathsf{V}_k$,
$\dim h(X) = \chi(X)$, , .

.

$$\dot\phi : M \otimes N \to N \otimes M, \quad \dot\phi = \oplus \dot\phi^{r,s}, \quad \dot\phi^{r,s} : M^r \otimes N^s \to N^s \otimes M^r$$

$\dot{\mathsf{M}}_k$. :

$$\phi : M \otimes N \to N \otimes M, \quad \phi = \oplus \phi^{r,s}, \quad \phi^{r,s} = (-1)^{rs} \dot\phi^{r,s}.$$

M_k, **正确的 motive 范畴**.

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$H_\ell, H_{\mathrm{dR}}, H_\sigma$.

+ 9%² k @™ O 8 T - C X 2 T ; f : X ! X - # f

i ˇf = 1! X – X ! B / f X – X = X X – ! 2 p 1 2 > Q (K 1; 1) = k:

ð Y x 2 > Q (K 1; X); y 2 > Q (K X; 1) - ¿ Ë ß ¡y 2 > Q (K 1; X –) -
[F + - “ • hx; ¡yi
x ¡y ¨ 2 p > Q (K 1; X X –) ! > Q (K 1; 1) ¦ X
£ š ¢ ¿ - © x y 2 > Q (K X; X) X

1 ” • ñ X = V– W - ð Y
f 2 > Q (K V; W) = > Q (K 1; X) -g 2 > Q (K W; V) - # cg
¡g A E W– y V— e È ç X êU… V y V— c ã V
ù È ç Ë ß Ë ß hf; cgi = i ` g f 2 > Q (K V; V) X

+ 9 Ó ƒ n ° • 0 § W ƒ ™ X - ÿ G . 2 H B ; M 2 @ãJ-B H M 2
¨ r 2 > Q (K h(X); h(X))

$$h^r(X) ! ^{B/} h^r(X); h^s(X) !^{0} h^s(X); s 6 r:$$

£ = i r = 2n r X n ¢

$$(h^r(X))_- = (h(X); \ _r)_- = (h(X); \ _{2n \ r})(n) = h^{2n \ r}(X)(n):$$

2 p Þ e Ë ß X

1 x Ê • ñ h(X) y h(Y)(i) Û - / H
f 2 $C^{n+i}_{>}$(X Y) -g 2 $C^{m}_{>}$ ^{i}(Y X) X
4 ž cg Ï ^{i}g 2 $C^{m}_{>}$ ^{i}(X Y) X{ x Ê n ¢

hf; ^{i}gi = i ` g f 2 > Q (K h(X); h(X)) :

ðY X #¨Ä'¿ F_q -•—öÕâÝ " èy -éÄ

$$](X(F_{q^n})) = h \; 6^n_q ; \; i = i\!\!\_` \; 6^n_q j h(X) :$$

¿ 6_q 6`Q#2M0BmXb

9© w2iáƒ UYÊè ,

$$Z(X;t) = 2tT \sum_{n \; 0} \frac{](X(F_{q^n}))}{n} t^n$$

$$= 2tT \sum_{n \; 0} \frac{1}{n} \; i\!\!\_` \; 6^n_q \; h(X) \quad t^n$$

$$= 2tT \quad i\!\!\_` \; H(1M \quad 6_q t) \; h(X)$$

$$= /2 \; i1 \quad 6_q t \; h(X) \quad ^{1} :$$

xÊ X

—V Š 8 T +» -¿Óƒ n- :V! V %²
cÁ XðY Ô«Ú¾"

$$(\ ) = /2i \quad V = {}^{n} \quad a_{n \ 1} {}^{n \ 1} + \quad + (\quad 1)^{n} a_{n} :$$

éÄ a_i = i` ^{i} ^ ^{i} V X

š¢ X

›¸gç" # /2i V = i` ^{n} ^ ^{n} V X
1¿• ñ -Ä

$$f_i = \overset{X}{\quad} (\quad {}^{i} \quad B/{}^{n} \quad {}^{i});$$

$$= S_n/(S_i \quad S_{n \quad i})$$

$$a_i = i` f_i \ \hat{}\ {}^{n} V :$$

• ˆ X

Q } V ¨ î ™ O 8 < Ä

i ` f i ^ n V = i ` i ^ i V / B K i n i :

š ¢ X

ï . 2 H B ; R M 2 N y G 2 K K 2 U X K ˆ " ® Y
i (B) = / B K ; i (A) ; : : : ; i (A i) J « Ú ¾ " X
ò ¢ V = 1 m -m 3 … ƒ ž / B K = m - U Y + N à š
ˆ " , ˆ - R ò Ú ¾ " © , ˆ X □

P – X

+ 9‰ ² Š 8 T + » V C ¿ c Á f 2 > Q (K ; V) -
ï • ˆ i = 1 Õ Ñ w ï 2 t T 1 y J e ± ñ
2 t T i ` f V = / 2 i 2 t T V :

x Ê X

—M ©%² ™ O 8 T + » -f : M ! M %² c Á -
é Ä Y ” ÿ è y ,

X (1)ⁿ i ` f ⁿ ^ ⁿ M tⁿ X i ` f ᵐ a v k ᵐ M tᵐ = 1:
n 0 m 0

š ¢ X

1 ¥3† ´ ƒ J • - é Q ¬ š ¢ + ` Þ 3 .. ƒ l - ® Ä
M M

^ ⁿ M a v k ⁿ M = ^ ⁿ M a v k ⁿ M:
0 n l 0 n l
n 2 p 2 M n Q / /

ð Y Y Ö ¥ U £ à š , > ¥ 0 Ö ` © V ,

0 ! ^ ˡ M ! ^ ˡ ¹ M a v k ¹ M ! ! a v k M ! 0;

• w ª Ë ß ¥ ~ ç F • g “ ã - Y š ¢ ©%² 3 ` ç X

š ¢ ŴÒ V X

+ 9 3 … ƒ n - “ $p_n = \frac{1}{n!}\sum_{2S_n}$; $q_n = \frac{1}{n!}\sum_{2S_n}(-1)^{b;M}$ -
ã ð + • a v Ꝁ y ^ ^{n} 3† ˆ « X
ð Y £

¿ X X Á 0 Ö ` ã ð $q_n p_m q_{n+1} p_m$ y $q_n p_{m+1} q_n p_m$ X
£ ± ñ [Ö ` ® 0 X
4 Ù - ï Ü + f ÿ = œ = [¨ , [%² @ ƒ e C ©3† ˆ X
Â e … Y – ò ,

– ò X

$$q_n p_m = \frac{n(m+1)}{m+n} q_n p_{m+1} q_n p_m + \frac{m(n+1)}{m+n} q_n p_m q_{n+1} p_m :$$

Ô ` Y x Ê ± ñ

P – X

—V ©î Š 8 + » - ©V c Á - é È `
x Ê = Ä " è y ,

$$/2i1 \quad t \; V^{1} = \sum_{n \; 0} i` \;^{n} \; a v ƘV \; t^{n}:$$

R ò U Y w 2 i á ƒ Ê

$$Z(X;t) = \sum_{n \; 0} \text{i̲`} \; 6_{q}\grave{}^{\,n} \; a v Ƙh(X) \; t^{n};$$

$$= \sum_{n \; 0} \text{i̲`} \; 6_{q}\grave{} \; h(a v ƘX) \; t^{n}:$$

U Y # K Q i B p B +á w 2 @ ƒ ¨%² 8 3† ´ ƒ ,

$$Z(X;t) = \sum_{n \geq 0} h(\text{a v } K^n X)\, t^n :$$

h Þ ñ ⊥` $h(X) = \sum_i (-1)^i$ i ` $h^i(X)$ -

n ¢ y n Ï È ç © , c X

4 - + ` Þ ê c Á $K_0(J_k)$! R - ® U Y ‹ ‘ ¿ w 2 i á ƒ X

ò " 2 i i β b^i = i ` $_i j h( )$ -

Ô « 0 ¿ > Q / ; 2 $h^{p;q}$ ˆ X

† ò : º i i b + ?̧”2 ,

+ 9%² 2n Ó W ƒ ™X - $P_{x;y}(X)$ n P
+ f > Q / ; 2 ÿ è á ƒ - /

$$P_{x;y}(X) = \sum_{n} \sum_{i;j} \sum_{n} h^{n+i;n+j}(X) x^i y^j :$$

—S %² ¼ µ -$S^{(n)} =$ a v K $S = S^n / S_n$ -
$S^{[n]}$ n P S n µ > B H #'2 ` iX
$h^{i;j} = h^{i;j}(S)$ - é Ä ˆ ” ,