DELOCALIZATION OF SCHRÖDINGER EIGENFUNCTIONS

NALINI ANANTHARAMAN

Abstract. A hundred years ago, Einstein wondered about quantization conditions for classically ergodic systems. Although a mathematical description of the spectrum of Schrödinger operators associated to ergodic classical dynamics is still completely missing, a lot of progress has been made on the delocalization of the associated eigenfunctions.

1. Some history

[illegible]

H ($H\phi = E\phi$, $E \in \mathbb{R}$),

$(x; t)$ (1).

ensembles

.1

2. High frequency delocalization

(M, g) d,

Δ M.

$L^2(M; \mathrm{Vol})$, Δ

$\longrightarrow +\infty$,

$(\phi_k)_{k\in\mathbb{N}}$ $L^2(M;$ $)$,

$\phi_k = -$ $_k\phi_k$.

M , ϕ_k ∂M ,

$\mathbb{R}$

$_k \longrightarrow +\infty$.

semiclassical limit

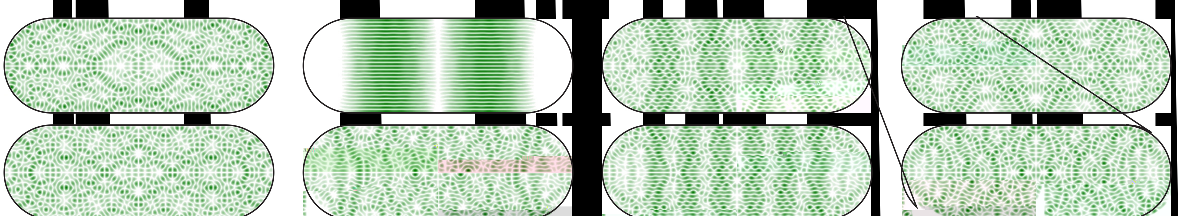

Figure 1. $|\phi_n(x;$ $= 3$ 9.

2.1. The geodesic flow . Δ $_k = -$

$\Delta\phi = E\phi$ $_k = E$ V

“ ”.

100 .

$-\hbar^2\Delta$ M

,

M ,

, fl . T^*M *cotangent bundle*

M . $(x;\ ) \in T^*M$ $x \in M$

(“ ” $\in T^*_x M$ (“ ” $(x;\ ) \in T^*M$,

$t \in \mathbb{R}$, $g^t(x;\ ) \in T^*M$,

$(g^t)_{t\in\mathbb{R}} : T^*M \longrightarrow T^*M$ fl ff g^{t+s} g^t g^s g^0

. fl

, *unit cotangent bundle* $S^*M = \{(x;\ ) \in T^*M; \|\ \|_x = 1\}$

g^t.

, ($_k \longrightarrow +\infty$),

$= i\Delta$

t

wave fronts . ,

$^{\epsilon}(x$

– ϕ (2),

($\epsilon \sim {}_{k}^{-1/2}$).

ffi ff

$e^{t\Delta}$, t

– (2) ( ) t

(2) .

2. L^p-norm as measures of delocalization ? fi

1

, ϕ_λ

$-\Delta\phi_\lambda =$, $\|\phi_\lambda\|_{L^2} = 1$ L^∞-

Theorem 1. *(known as Hörmander's bound)*

$$\|\phi\|_\infty = O({}^{(d-1)/4}):$$

, . L^p , $\le p$ $+\infty$

Theorem 2 (10).

$$\|\phi_\lambda\|_{L^p} = O({}^{\frac{(p)}{2}})$$

where

- $(p) =$ $\left(\frac{1}{2} \quad \frac{1}{p}\right) - \frac{1}{2}$ *for* $\frac{2(d}{(d} \le p \le +$;
- $(p) =$ $\frac{-1}{2}$ $\left(\quad - \frac{1}{p}\right)$ *for* $2 \le$ $\le \frac{2(d+1)}{(d-1)}$.

M. d

M $p_c = \frac{2(\ \ +1)}{(\ \ -1)}$.

$\mathbb{S}^d$ $p \ge p_c$ (

), $p \le p_c$ (

). , L^p

. $p >$

L^p-

",

$x \in M$, $\mathcal{L}_x$ S_x^*M x, . .

$$\mathcal{L}_x = \{ \quad \in S_x^*M, \exists t > 0; g^t(x; v) \in S_x^*M\}$$

${}_x$ S_x^*M.

Theorem 3 (, 10). • *Assume there exists a subsequence* ${}_{n_k} \longrightarrow +\infty$ *and* $C > 0$ *such that* $\|\phi \|_\infty \ge C {}^{\frac{(\infty)}{2}}$. *Then there exists such that* ${}_x(\mathcal{L}) > 0$;

- *If* M *is real analytic, the existence of such subsequence* ${}_{\lambda_{n_k}}$ *is to the existence of* x *such that* ${}_x$ $S M$, *and the first return map* ${}_x : S_x^*$ $\longrightarrow S_x^*M$ *possesses an absolutely continuous invariant probability measure. (Moreover, in* *that case, there exists* $t_0 > 0$ *such that* $g^{t_0}(x;v) \in S_x^*M$ *for all* $v \in S_x^*M$*, that is, there is a common return time).*

- *If* M *is real analytic and* $\dim M = 2$*, the existence of such subsequence* $\phi_{\lambda_{n_k}}$ *is* [illegible] *to the existence of* $x \in M$ *and* $t_0 > 0$ *such that* $g^{t_0}(x;v) = (x;v)$ *for all* $v \in S_x^*M$*.*

[illegible]

Anosov property, [illegible]

[illegible] 3

$$\|\phi_\lambda\|_{L^p} = o\left(\lambda^{\frac{\delta(\infty)}{2}} \right) \text{ [illegible] } 1 \text{ [illegible] } o).$$

[illegible]

Theorem 4. *(i) (* [illegible] 1 [illegible] *) If* $d = 2$ *and* M *has no conjugate points, or if* $d \geq 2$ *and* M *has non-positive sectional curvature, for* $p = +\infty$*,*

$$\|\phi_\lambda\|_{L^p} = O\left(\frac{\lambda^{\frac{\delta(p)}{2}}}{\sqrt{\log \lambda}} \right):$$

(i') ([illegible] 30 *) Statement (i) also holds if* M *has no conjugate points, for all* $d \geq 2$*.*

(ii) ([illegible] 3 *) (i) holds for all* $p > p_c$*.*

(iii) ([illegible] 2 , 22 *) If* M *has non-positive sectional curvature, for* $p < p_c$*, there exists* $\sigma(p;d) > 0$ *such that*

$$\|\phi_\lambda\|_{L^p} = O\left(\frac{\lambda^{\frac{\delta(p)}{2}}}{(\log \lambda)^{\sigma(p;d)}} \right)$$

(iv) ([illegible] 23 *) Statement (iii) still holds for* $p = p_c$*.*

[illegible]

3. The Shnirelman theorem and the Quantum Unique Ergodicity conjecture.

[illegible] $|\phi_k(x)|^2 d\mathrm{Vol}(x)$.

[illegible] $k \longrightarrow +\infty$) [illegible] $|\phi_k(x)|^2 d\mathrm{Vol}(x)$

“ ” ([redacted]). [redacted] -
[redacted] fi [redacted] fl ,

, [redacted] $a : S^*M \longrightarrow \mathbb{R}$, -
$(x_0;$ [redacted] $0) \in$ [redacted] $^*M$, [redacted] $\frac{1}{T}\int_0^T a \circ g^t(x_0;\ _0)dt$ [redacted] $T \longrightarrow$ [redacted] ∞
[redacted] $\int_{S^*M}$ [redacted] dL [redacted] L [redacted] S^*M
(. . [redacted]), [redacted]
[redacted] T [redacted] M

Quant[redacted]m [redacted]r[redacted]dicity T[redacted]e[redacted]rem[redacted]n[redacted]elman [redacted]heorem).

Theorem [redacted] 5 [redacted] , [redacted] 10 , 0, 11 *Let* ([redacted] ; *g* [redacted] *be a compact R*[redacted]*em*[redacted]*n*[redacted]*ar*[redacted] *manif*[redacted]*d*[redacted] *with* [redacted] *me*[redacted]*ric norm*[redacted]*alized so that* Vol[redacted]$M) =$ [redacted] *all* Δ *the Lapla*[redacted]*-B*[redacted]*tr*[redacted]*mi operat*[redacted]*r* [redacted] *n* M [redacted] *ume that th*[redacted] *geodesic flow of* [redacted] M *is* [redacted] *with respect to* [redacted] *he* [redacted] *io*[redacted] *ille mea*[redacted]*ure*[redacted] *Let* [redacted] *∈*[redacted] *be an or*[redacted]*honormal basis o*[redacted] $L^2(M;$ [redacted] *m*[redacted]*ade of eigenfunct*[redacted]*ons*[redacted] *of*[redacted] *e Lapla*[redacted]*ar*[redacted]

$$\Delta [redacted] = - [redacted] _k [redacted] _k \leq [redacted] _k [redacted] \longrightarrow +\infty:$$

Let a *be a* [redacted] *con*[redacted] *n*[redacted] *u*[redacted] *funct*[redacted] *n*[redacted] *n* M [redacted] *er*[redacted]

(3) [redacted] 1 [redacted] $\sum_{k,\lambda_k \leq}$ [redacted] $\langle$ [redacted] $; a\phi$ [redacted] $-\int_M a(x$ [redacted] $d\,\mathrm{Vol}(x)\Big|^2 \underset{\lambda \longrightarrow +\infty}{\longrightarrow}$ [redacted]

where the [redacted] *or*[redacted] *a*[redacted] *i*[redacted] *facto*[redacted] *is* [redacted] ([redacted] k [redacted] $_k \leq$ [redacted] $\}|$[redacted]

$$\langle [redacted] ; [redacted] _k [redacted] {}^2(M) [redacted] a(x [redacted]) [redacted] d\,\mathrm{Vol}(x) [redacted]$$

Remark [redacted] *e*[redacted] *e*[redacted] *aro li*[redacted] *t* [redacted] *3) i*[redacted] *s* [redacted] *at there* [redacted] *exists a subset* S [redacted] $\subset \mathbb{N}$ *o*[redacted] *e*[redacted] *ity* 1 *such that* [redacted]

( ) [redacted] $\langle$ [redacted] $; a\phi_k\rangle$ $_{n}$[redacted]$_{,n\in S} \int_M a(x$ [redacted] $d\,\mathrm{Vol}(x):$

In addition[redacted] *us*[redacted] *e fa*[redacted] *that the* [redacted] *e of contin*[redacted] *ous functions is* [redacted] *parable* [redacted] *one can actually find* $S \subset$ [redacted] *f* [redacted] *density* 1 *such* [redacted] *t* ( ) *holds* [redacted] *r* [redacted] $a \in C^0(M$ [redacted] *In oth*[redacted] *r words, the sequence of m*[redacted] *res* $(|\phi_k(x)|^2 d\,\mathrm{V}$[redacted]$)_{n\in S}$ *conve*[redacted]*rges weakly to th*[redacted] *uniform* [redacted] *measure* $d\,\mathrm{Vol}(x)$.

Actually, the full [redacted] *atement of the theorem says that there exists a subset* $S \subset \mathbb{N}$ *of density* 1 *such that*

( ) [redacted] $$\langle \phi_k; A\phi_k\rangle \underset{n\longrightarrow +\infty, n\in S}{\longrightarrow} \int_{S^*M} {}^0(A)dL$$

for every pseudodifferential operator A *of order* 0 *on* M. *On the right-hand side,* ${}^0(A)$ *is the principal symbol of* A, *that is a function on the unit cotangent bundle* S^*M. *Equation 4 corresponds to the case where* A *is the operator of multiplication by the function* a.

[illegible]

[illegible]

[illegible]

“ ” ϕ_{n_k}.

(ϕ_{n_k}). ,

, $g^t_\sharp =$, $t \in \mathbb{R}$.

$\longrightarrow +\infty$,

2.1.

Theorem 7. 3 *Assum M is a compact R emannian manifold ith negative sectional cur ature. Assume $\langle \phi_{n_k}, A\phi_{n_k} \rangle$ converges to $\int_{T^*M} \sigma^0(A) d$ for all A. Then has positive ent opy.*

$h_{KS}( )$,

- $h_{KS}( ) = 0$
- $\mapsto h_{K}( )$ $h_{KS}(_1 + (1 -) _2) = h_{KS}(_1) + (1 -) h_{K}$, $_1$, $_2$, $\in [0, 1]$
- (-M) ,

$$( ) \qquad h_{KS}( ) \leq \int \left(\sum_{j=} ( ) \right) d ( )$$

$_j^+( )$

- ,
-) L , 2)
- , d (M $2d-1$), $d-1$, ,

$$( ) \qquad h_{KS}) \leq d$$

Corollary 1. *$\in S^*M$ be the unic ll p t ing a periodic ajectory on the geodes flo al M has negat vat re countably many periodic geodesics) Let in Theorem 7. Th) <*

, , *scar* (,).

, *strong scarred*

[redacted] *partial scar*, [redacted] $(\Gamma) > 0$ [redacted]

Cor[redacted]**lary 2.** *The su*[redacted]*port of* [redacted]*as Hausdorff dimension >* [redacted]

≥ 1 [redacted]

The[redacted]**em 8** ([redacted]). *Assume* [redacted] *is* [redacted] *compact Riemannian man*[redacted]*old of di*[redacted]*ensio*[redacted] *with co*[redacted]*sectio*[redacted]*al curvature* $-$ [redacted] *has entropy greater than* [redacted]

[redacted] $d-1$ [redacted]

1=2. [redacted] $\frac{d-1}{2}$

Cor[redacted]**ary 3.** [redacted]

[redacted] T[redacted]M [redacted] d, [redacted]

Cor[redacted] **4.** [redacted] *be* [redacted] *ion o*[redacted]*all po*[redacted]*nts ly*[redacted] *a* [redacted]*sed traje*[redacted]*ory on the geod*[redacted] *in* [redacted] *7,* [redacted]*th M of co*[redacted] *neg*[redacted]*ive curva*[redacted]*re. Then* [redacted]

[redacted] $1 - $ [redacted] $_2$, [redacted] Γ,

h_{KS} [redacted] $\geq \frac{d-1}{2}$

[redacted] $d-1$, [redacted] $\leq 1{=}2.$

[redacted] $\sum_{j=1}^{d-1}$ [redacted]

[redacted] $1{=}2$ [redacted]

Th[redacted]**eor**[redacted]). [redacted] *l su*[redacted]*por*[redacted]*at* [redacted] $(\Omega) > 0$ *for any* [redacted]*on-empty op*[redacted]

2. . Some questions on non-compact manifolds.

Absolutely continuous spectrum.

Large ... oscillations ... non-compact ...

M … $|\phi_\lambda(x)|^2 d\mathrm{Vol}(x)$ … $|\phi_\lambda(x)|^2 d\mathrm{Vol}(x)$ … 112 …

… $C_c^\infty(M)$ …

… scale delocalization …

… *spectral statistics* …

… *integrable dynamics* … *chaotic* …

… 21 …

… $\Delta =$ … $+\infty$ … 1 …

regular

3.

3.1. Overview of the problem. $G = (V, E)$.

localized, *delocalized*

$\subset \mathbb{R}$,

- *spectral localization :* I,
- *exponential localization:*
- *dynamical localization* I

- *spectral delocalization* I,
- *ballistic transport*

spatial delocalization.

$(G$

$(G$ $N \to \infty$

N, $(\phi_j)_{j=1}^N$

ϕ_j

$\sum |\phi_j$

N

- *ℓ^∞ norms*

  $\|\phi$ $1/\sqrt{N}$
- *ℓ^p norms:* ϕ_j $1/p$ $1/2$

  $f(p)$ $1/p - 1/2$.
- *S* $\sum_{x \in}$ $\geq 1 -$

  $(\ )_{x \in \Lambda}$ ϵ Λ
- *Quantum ergodicity* $\{\ldots\}$
  $\sum |\phi_j(x)|^2$ $a($

most j. [redacted] *all* j, [redacted] *quantum unique ergodicity.*

[redacted] , ... [redacted] *almost sure* [redacted] ℓ^∞- [redacted] *random* [redacted] *rand*[redacted]

2, [redacted] , 13, 1 [redacted] .1. [redacted] ff [redacted]

[redacted] 3 [redacted]

[redacted] $(q+1)$-

[redacted] $q+1$ [redacted] $(\ldots_N)_{N\in\mathbb{N}}$ [redacted] $(V_N; E_N)$ [redacted] V_N

$$\mathcal{A}_N [redacted] \sum_{x\sim y} [redacted]$$

[redacted] x [redacted] y [redacted]

$$[redacted] f(x) [redacted] f(y) [redacted] x)):$$

[redacted] $\mathcal{A}_N$ [redacted] $I =$ [redacted]

[redacted] $(G_N$ [redacted] *a se*[redacted]*ence o*[redacted]*)-reg*[redacted]*r graphs* [redacted] V_N [redacted] $N)$ [redacted] *that*[1] [redacted] *exists* c [redacted] > 0 [redacted] $k \leq c$ [redacted] N, [redacted] *any* [redacted] $y \in$ [redacted]

[redacted] $|\{$*pat*[redacted]*of* [redacted]*th* k [redacted] $y\}|$ [redacted] $q^{k(\frac{1-}{2}}$ [redacted]

[redacted] *if* ϕ [redacted] *the dis*[redacted] *and* [redacted] $\Lambda \subset V_N$ [redacted] *that*

$$[redacted] |\phi(x)|^2 \geq [redacted] \sum_{\in V_N} |\phi([redacted] |^2;$$

th[redacted] $|\Lambda| \geq$ [redacted] *— where* [redacted] > 0 *is given as a*[redacted]*explicit* [redacted] *of* ϵ; [redacted]

[redacted]

$$H_N(\phi) = -\frac{1}{\log N} \sum_x |\phi(x)|^2 [redacted] |\phi [redacted] ^2$$

[1] This assu[redacted]tion h[redacted]s in particular if th[redacted]injectivity radius is $\geq$ [redacted] $\ln N$. The interest of the weaker assumption is[redacted]at it h[redacted]ds for typical random[redacted]regular graphs [90].

L^∞- ... $\|\phi\|_\infty = O((\log N)^{-1/...})$.

... =4 ...

(11) ...

3.3. QE on ... egular graph...

... fi ...

... $G_N = (V_N; E_N)$ ... $1)$- ...

(EXP) ... > 0 ... $(q+1)^{-1}\mathcal{A}_N$ ... $\ell^2(V_N$ ... $1\} \cup [-1+\ ;1-\ ]$ ... N.

... G_N ... $\mathcal{A}_N$ ...

0, ... (... 3).

(B ST) ... R,

$$\frac{|\{x \in V_N; \ldots R\}|}{\ldots} \underset{N \to \ldots}{\longrightarrow} 0$$

x ... r

... (q + ...)- ... fi ... $(q+1$

... **(B ST** ...

$(\ldots^{(N)}; \ldots ; \ldots)$ ... G_N ...

$\subset \mathbb{R}$

$$\ldots N \ldots |\{j; \ldots \in \ldots\}| \underset{N \to +\infty}{\longrightarrow} \int \ldots m(\ )d$$

$_o$ ... $\mathfrak{X}$).

(12) $$\int \ ^k m(\ )d\ = \langle\ _o; \mathcal{A}^k_{\mathfrak{X}}\ _o\rangle_{\ell^2(\mathfrak{X})}$$

... $\mathfrak{X}$ ... $\mathfrak{X}$ ... o

... o ... x ... m ...

... $(-2\sqrt{q}; 2\sqrt{q}$ ...

... $^{(N)}_j$ ... $(-2\sqrt{q}; 2\sqrt{q})$,

...); q ...

11.

Theorem 11 ([illegible]). *Let $(G_N) = (V_N, E_N)$ be a sequence of $(q+1)$-regular graphs with* $|V_N| = N$. *Assume that (G_N) satisfies* **(BST)** *and* **(EXP)**.

Let $(\phi_1^{(N)}, \ldots, \phi_N^{(N)})$ be an orthonormal basis of eigenfunctions of $\mathcal{A}_N$ in $\ell^2(V_N)$.

Let $a_N : V_N \longrightarrow \mathbb{C}$ be a sequence of functions such that $\sup_N \sup_{x \in V_N} |a_N(x)| \leq 1$. Define $\langle a_N \rangle = \frac{1}{N} \sum_{x\in V_N} a_N(x)$.

Then

$$\frac{1}{N}\sum_{j=1}^{N} \Big| \langle \phi_j^{(N)}, a_N \phi_j^{(N)} \rangle_{\ell^2(V_N)} - \langle a_N \rangle \Big|^2 \underset{N\longrightarrow+\infty}{\longrightarrow} 0.$$

[illegible] > 0

(13) $$\frac{1}{N}\Big|\Big\{ j \in [1, N], \ \Big| \langle \phi_j^{(N)}, a_N \phi_j^{(N)} \rangle_{\ell^2(V_N)} - \langle a_N \rangle \Big| > [illegible] \Big\}\Big| \underset{N\longrightarrow+\infty}{\longrightarrow} 0.$$

$\langle \phi_j^{(N)}, a_N \phi_j^{(N)} \rangle_{\ell^2(V_N)}$ [illegible] $\phi_j^{(N)}$ [illegible] $a_N \phi_j^{(N)}$, [illegible] $\sum_{x\in V_N} a_N(x)|\phi_j^{(N)}(x)|^2$. [illegible] 11 [illegible] V_N,

$$\sum_{x\in V_N} |\phi_j^{(N)}(x)|^2 \ [illegible] \ \frac{1}{N}\sum_{x\in V_N} \ [illegible]$$

[illegible] a_N [illegible]. (13) [illegible] N [illegible] j, [illegible]

3. . Non-regular graphs: from spectral to spatial delocalization.

[illegible] *regular* [illegible] *quantum ergodicity theorem* [illegible] *spectral delocalization* [illegible] *spatial delocalization* [illegible]

"If a large finite system is close (in the Benjamini-Schramm topology) to an infinite system having purely absolutely continuous spectrum in an interval I, then the eigenfunctions (with eigenvalues lying in I) of the finite system satisfy quantum ergodicity."

$(G_N)_{N\in\mathbb{N}}$ [illegible] 3 [illegible] V_N [illegible] E_N [illegible] G_N, [illegible] $|V_N| = N$ [illegible] $N \longrightarrow \infty$. [illegible] $\mathcal{A}_N : \mathbb{C}^{V_N} \longrightarrow \mathbb{C}^{V_N}$

$$(\mathcal{A}_N f)(v) = \sum_{w\sim v} f(w)$$

v [illegible] w [illegible] v [illegible] $\mathcal{A}_N$, [illegible] $N \longrightarrow +\infty$.

(EXP) (G_N) .

$\mathbb{N}$, , fi $P_N : \mathbb{C}^{V_N} \longrightarrow \mathbb{C}^{V_N}$

$$(P_N f)(x) = \frac{1}{d_N(x)} \sum_{y \sim x} f(y);$$

$d_N(x)$ x. **(EXP)**

P_N , , 1 P_N ,

P_N $[-1 + \quad 1 \quad] \cup \{1\}$, > 0 N.

1 , .

G_N .

P

(, 3).

(G_N)

(BST) > 0,

$$\lim_{N \to \infty} \frac{|\{x \in V_N : \quad_{G_N}(x) < r\}|}{N} = 0;$$

$_{G_N}($ *injectivity radius* x, $B_{G_N}(x, \quad)$

.

( ),

(G_N),

rooted graphs $\mathbb{N}$,

(B .

(BSCT) *tree* $[\mathcal{T}; o]$

$\{[\mathcal{T}; o]\}$ $\mathbb{P}$

$\mathcal{A}_N$ $\ell^2(V_N)$. **(BSCT)**

$: \mathbb{R} \longrightarrow \mathbb{R}$,

$$\sum_{j} \longrightarrow_{N \longrightarrow +\infty} \quad {}_o; \quad (\mathcal{A}_\mathcal{T}) \quad {}_o \rangle =: \int \quad dm(t),$$

$\mathcal{A}_\mathcal{T}$ $\mathcal{T}$ $(\mathcal{T})$.

$\mathbb{P}$. *integrated density of states*

.

I

$\mathbb{P}$- $[\mathcal{T}; o]$

$\mathbb{P}$). x $\in \mathbb{C} \setminus \mathbb{R}$

$$(x; y) = (\mathcal{A}_\mathcal{T} \quad)^{-1} \quad {}_{\ell^2(\mathcal{T})}$$

$v, w \in \mathcal{T}$ [redacted] $v \sim w$, [redacted] $\mathcal{T}^{(v|w)}$ [redacted] $\mathcal{T}$ [redacted] v [redacted] w. [redacted] $\mathcal{A}_{\mathcal{T}^{(v|w)}}$ [redacted] $\mathcal{G}^{(v|w)}(\cdot,\cdot;\ )$ [redacted] $_w^{\gamma}(v)$ [redacted] $= -$ [redacted] $^{(v|w)}(v,$ [redacted] $;\ )$.

(Green) [redacted] I, [redacted] $s > 0$

$$\sup_{\lambda \in I, \ \in (0,1} \ \mathbb{E}\left(\sum_{y: y\sim o} |\operatorname{Im} \ _o^{\lambda+i\eta_0}(y)|^{-s}\right) < \infty.$$

[redacted] **(Green)**, fi [redacted] ([redacted]) [redacted]

$[\mathcal{T}, o]$

(1) [redacted] $J = \langle \ ; \mathbb{1}_J(\mathcal{A}_{\mathcal{T}})\ _o\rangle$ [redacted] $J \subseteq \mathbb{R}$.

[redacted] **(Green)** [redacted] $\sup_{\lambda\in I, \eta_0>0} \mathbb{E}(|\mathcal{G}$ [redacted] $o, o)$ [redacted] $) <$ [redacted] ∞.

[redacted] $\mathbb{P}$- . . $[\mathcal{T}, o]$, [redacted] $_o$ [redacted] -

[redacted] I [redacted] $\frac{1}{\pi}\operatorname{Im}\mathcal{G}$ [redacted] $^{+i0}(o, o)$. [redacted] , **(Green** [redacted] $\mathbb{P}$- . .

$\mathcal{A}_{\mathcal{T}}$ [redacted] I. [redacted] -

[redacted] 1 [redacted]

[redacted] **Gre** [redacted] **n)**, [redacted] < 0, [redacted]

[redacted] , ,

[redacted] - [redacted] o [redacted]

I [redacted] **(** [redacted] **reen)**, [redacted] fi [redacted]

[redacted] $\bar{I}_1$ [redacted] I. [redacted] G_N [redacted] Γ_N [redacted]

([redacted] $_N)$ [redacted] $; \tilde{y}$ [redacted] $\in \mathbb{C}$ [redacted]

[redacted] $\widetilde{\mathcal{A}}_N$ [redacted] $\widetilde{G_N}$

(1 [redacted] $(\tilde{x}$ [redacted] $=$ [redacted] $_{\tilde{x}}, ($ [redacted] $_N -\ )^{-1}$ [redacted] $\rangle_{\ell^2(\widetilde{G_N})}$.

Th [redacted] **ore** [redacted] **2** ([redacted]). *Assume* [redacted] *t* (G_N, W_N) [redacted] *isfies* [redacted] ***BS*** [redacted] ***T),*** ***(E*** [redacted] ***P)*** [redacted] ***(G*** [redacted] ***een*** [redacted]

[redacted] *ll* ([redacted] $)_{j=1}^{N}$ [redacted] *e* [redacted] *ue* [redacted] *of* [redacted] *o* [redacted] $\ell^2(V_N)$, [redacted] *d let* $(\phi_j^{(N)})_{j=}$ [redacted] *e a c* [redacted] *spo* [redacted] *ding orth* [redacted] *no* [redacted] *eigen* [redacted] *as* [redacted]

[redacted] *r e* [redacted] *h* [redacted] *, let* [redacted] $=$ [redacted] *a* [redacted] *ncti* [redacted] *on* [redacted] $_N$ *with* $\sup_N \sup_{x\in V_N} |a_N$ [redacted] $| \le 1$

[redacted] *en*

$$\lim \ \lim_{N\to\infty} \ \left| \sum_{x \in V_N} a(\ )|\phi_j^{(}\ (x)|^2 - \sum_{x} a(x) \ _{\lambda_j^{(N)}+i\eta}^{N} \right| = 0$$

for [redacted] *om* [redacted] *fa* [redacted] *ily* [redacted] *proba* [redacted] *asu* [redacted] *s* [redacted] *on* V_N, [redacted] *dexed by a pa* [redacted] *ameter* [redacted] $\mathbb{C}$ [redacted] $\mathbb{R}$, *def* [redacted] *ed* [redacted] *llows*

$$(x) \ \ \frac{\operatorname{Im}\tilde{g}_N^{\gamma}(\tilde{x}, \tilde{x}}{\sum_{y\in V_N}\operatorname{Im}\tilde{g}_N^{\gamma}(\ \tilde{y})}.$$

Here, $\tilde{x} \in \widetilde{G_N}$ is a lift of $x \in V_N$.

, $\epsilon > 0$,

$$(1\ \ ) \quad \frac{1}{N}\left|\left\{\ {}^{(N)}_{j} \in I_1 : \left|\sum_{x\in V_N} a(x)|\phi_j^{(N)}(x)|^2 - \sum_{x\in V_N} a(x)\ {}^{N}_{\lambda_j^{(N)}+i\eta_0}(x)\right| > \epsilon\right\}\right| \underset{N\to+\infty,\,\eta_0\downarrow 0}{\longrightarrow} 0:$$

12 [redacted] ${}^{N}_{\gamma}$

. [redacted] $a_N = \mathbb{1}_{\Lambda_N}$,

$\Lambda_N \subset V$ [redacted] $\gamma \approx N$ [redacted] $\in (0;1)$. [redacted] (G_N) ,

$\widetilde{G_N}$ [redacted] N ([redacted] $(q+1)$- [redacted])

$\tilde{g}_N^\gamma(\tilde{x};\tilde{y})$ [redacted] $\mathcal{G}^\gamma(\tilde{x};\tilde{y})$ [redacted] . M [redacted] ,

$\mathcal{G}^\gamma(\tilde{x};\tilde{x}) = \mathcal{G}^\gamma(o;o)$ [redacted] $\tilde{x} \in \widetilde{G_N}$. [redacted] ${}^{N}_{\gamma}$

V [redacted]). [redacted] ( ) [redacted] $\mathbb{1}_{\Lambda_N}\phi_j^{(N)}\|^2 \approx$ [redacted] $\phi_j^{(N)}$.

$\phi_j^{(N)}$ [redacted] , [redacted] $\Lambda_N \subset V_N$

[redacted] fi [redacted] $|\phi_j^{(N)}|^2$.

, [redacted] ${}^{N}_{\gamma}(\Lambda_N) =$ [redacted] $|\Lambda_N|$ [redacted] N. ,

$c_\alpha > 0$ [redacted] $\Lambda_N \subset V_N$ [redacted] $|\Lambda_N| \geq$ [redacted] ,

$$(1\ \ ) \quad \text{[redacted]} \liminf_{N\to\infty} \inf_{\lambda\in I} \text{[redacted]}_{+i\eta_0}(\Lambda_N) \geq 2c_\alpha:$$

[redacted] (1),

Co[redacted]**ry 1**[redacted]. *For any* [redacted] *1), there e*[redacted]*ts $c_\alpha > 0$ such tha*[redacted]*r any $\Lambda_N \subset V_N$ with $|\Lambda_N|$* [redacted] *N,* [redacted] *have*

$$\frac{1}{N}\text{[redacted]} \in I : \left\|\mathbb{1}_{\Lambda_N}\text{[redacted]}\right\|^2 < c_\alpha\right\} \underset{N\to+\infty}{\longrightarrow}\text{[redacted]}$$

[redacted] $\mathbb{1}_{\Lambda_N}\phi_j^{(N)}\|^2 \approx$ [redacted] $\phi_j^{(N)}$, [redacted]

[redacted] $\phi_j^{(N)}\|^2 \geq$ [redacted] > 0 [redacted] $\phi_j^{(N)}$ [redacted]

[redacted] γ [redacted]

[redacted] $F : \mathbb{R} \to \mathbb{R}$, [redacted] $\in I$,

$$(2\ \ ) \quad \text{[redacted]} \frac{1}{N}\sum_{x\in V_N} F\left(\text{[redacted]}(x)\right) \underset{N\to+\infty}{\longrightarrow} \text{[redacted]}\left(F\left(\frac{\operatorname{Im}\mathcal{G}^{\lambda+i\eta}\text{[redacted]}}{\mathbb{E}(\operatorname{Im}\mathcal{G}^{\lambda+i}\text{[redacted]}))}\right)\right):$$

[redacted] $N\ {}^{N}_{\lambda+i\eta_0}(x))$ ([redacted]

V_N [redacted] $\left(\frac{\operatorname{Im}\text{[redacted]}(o,o)}{\mathbb{E}(\operatorname{Im}\text{[redacted]}(o,o))}\right)$. [redacted]

(x) [redacted] $1/N$ [redacted] N, [redacted] .

R[redacted]**ark 14.** [redacted]*he* [redacted]*sults* [redacted]*rove*[redacted]*in* [redacted] *actua*[redacted]*y hold fo*[redacted]*more ge*[redacted]*Sch*[redacted]*ödinger*[redacted]*erators th*[redacted]*adjacen*[redacted]*ma*[redacted]*rices* [redacted] *one can conside*[redacted]*weighte*[redacted] *Laplaci*[redacted]*th*[redacted]*conducta*[redacted]*ces on th*[redacted]*ges) an*[redacted]*add a po*[redacted]*tial, in other w*[redacted]*ds, on e*[redacted]*ch G_N,* [redacted] *nsider a* [redacted]*iscrete S*[redacted]*dinger* [redacted]*erat*[redacted] *$\mathcal{H}_N$* [redacted] *The*[redacted]*imiting obj*[redacted]*t in ass*[redacted]*mption* **(BS**[redacted]**)** [redacted]*s now a* [redacted]*andom* *rooted tree $[\mathcal{T}; o]$ endowed with a random Schrödinger operator $\mathcal{H}$. Assumption* ***(Green)*** *has to be modified, replacing the adjacency matrix $\mathcal{A}$ by the operator $\mathcal{H}$. Similarly, in the statement of the theorem, the Green functions $\tilde{g}_N^{\gamma}$ to be considered are those of $\mathcal{H}_N$ lifted to the universal cover $\widetilde{G_N}$.*

Remark 15. *In particular, our result applies to the case where the limiting system $([\mathcal{T}; o]; \mathcal{H})$ is $\mathcal{T} = \mathfrak{X}$ (the $(q+1)$-regular tree) with an arbitrary origin o, and $\mathcal{H} = \mathcal{H}_\epsilon = \mathcal{A} + \epsilon\mathcal{W}$ where $\mathcal{W}$ is a random real-valued potential on $\mathfrak{X}$. More precisely the values $\mathcal{W}(x)$ $(x \in \mathfrak{X})$ are i.i.d. random variables of common law . This is known as the Anderson model on $\mathfrak{X}$. It was shown by A. Klein that the spectrum of $\mathcal{H}_\epsilon$ is a.s. purely absolutely continuous on $I = (-2\sqrt{q} + \;; 2\sqrt{q} - \;)$, provided ϵ is small enough (depending on). This just assumes a second moment on . Under stronger regularity assumptions on , one can show that Assumption* ***(Green)*** *holds on I (see 11 , following Aizenman-Warzel 2). Examples of sequences of expander regular graphs G_N with discrete Schrödinger operators $\mathcal{H}_N$ converging to $([\mathfrak{X}; o]; \mathcal{H}_\epsilon)$ are given in 10 .*

Remark 16. *Examples of sequences of - satisfying our three assumptions were investigated in 11 . In the examples considered there, the limiting trees $\mathcal{T}$ are fi ; roughly speaking, those are trees where the local geometry can only take a finite number of values. If $\mathcal{A}$ is the adjacency matrix of such a tree, we showed in 11 that the spectrum of $\mathcal{A}$ is a finite union of closed intervals, and that there are a finite number of points $y_1; \ldots; y_\ell$ in such that Assumption* ***(Green)*** *holds on any I of the form $\setminus ([y_1 - \;; y_1 + \;] \cup \ldots \cup [y_\ell - \;; y_\ell + \;])$ (for any > 0). We showed – extending Remark 15 – that on such trees, Assumption* ***(Green)*** *remains true after adding a small random potential to $\mathcal{A}_\mathcal{T}$. Finally, we showed the existence of sequences (G_N) converging to $\mathcal{T}$ and satisfying the* ***(EXP)*** *condition.*

. Perspectives and link with other work

.1. Random regular graphs. 11

deterministic (G_N) *a*.

“ ” “ ” *a*.

random o

random fi 11 , 1 , 13

fi ( )

N $\{1; \ldots; \}$

$(q+1)$-

($(q+1)N$

fi .

1 , 13, “ ” *adja-*

cency mat $a_N : \{1; \ldots; N$ $\mathbb{R}$,

$(q+1)$- ... $\{1;\dots;N\}$ ... $\sum_{x=1}^N a_N(x)|\phi_j^{(N)}(x)|^2$ ... $\langle a_N\rangle$ ... *all* ... j, ...

Theorem 17 (... 1). *Let* ω *be such that* $\sqrt{q} \geq (\omega ... 1)2^{2\omega+45}$.

(i) With probability $\geq 1 - o(N^{-\omega+8})$ *on the choice of the graph*

$$\|\phi_j\|_\infty \leq \frac{(\log N)^{121}}{\sqrt{N}}$$

for all eigenfunctions associated to eigenvalues such that $| \ _j \pm 2\sqrt{q}| > (\log N)^{-3/2}$.

(ii) (Quantum Unique Ergodicity for random regular graphs) Given an observable $a_N : \{1;\dots;N\} \longrightarrow \mathbb{R}$, *we have, with probability* $\geq 1 - o(N^{-\omega+8})$ *on the choice of the graph, for* N *large enough,*

$$\left|\sum_{x=1}^N a_N(x)|\phi_j^{(N)}(x)|^2 - \langle a_N\rangle\right| \leq \frac{(\log N)^{250}}{N}\sqrt{\sum_x |a_N(x)|^2}; \tag{21}$$

for all eigenfunctions associated to eigenvalues $\ _j \in (-2\sqrt{q}+\epsilon; 2\sqrt{q}-\epsilon)$ *(bulk eigenvalues).*

In particular, if $a_N = 1_{\Lambda_N}$ *where* $\Lambda_N \subset \{1;\dots;N\}$, *we find*

$$\left|\sum_{\Lambda_N} |\phi_j^{(N)}(x)|^2 - \frac{|\Lambda_N|}{N}\right| \leq \frac{(\log N)^{250}}{N}\sqrt{|\Lambda_N|}$$

(... > 8 ... $q > 2^{128}$).

... ℓ^∞- ... , ...

... $|\Lambda_N| > (\log N)^{500}$. ... 11, ...

∞, ...

$$\left|\sum_{\Lambda_N} |\phi_j^{(N)}(x)|^2 - \frac{|\Lambda_N|}{N}\right| \ ... \ \frac{1}{\sqrt{N}\log N}\sqrt{|\Lambda_N|}$$

...),

most ... j ... $|\Lambda_N| > \ ...^{1/2}$ ... , ... ,

... a_N ... 11 ...

(1), a_N ...

... a_N ... (

...),

...),

... 1 ... , ... almost sure ...

Remark 18. *Note that we emphasize Theorem 17 from* 1 *because our main concern here is the delocalization of eigenfunctions. The main focus of* 1 *is however on the universality of the local spectral statistics for random regular graphs. This would deserve a separate paper.*

... 2 ... , ...

... G_N ... N ... , ... $\phi_j^{(N)}$,

$\sqrt{N}\phi_j(x)$ x $\{1, \dots, N\}$
$\mathcal{N}(0; \ {}^2_j)$ $0 \leq \ _j \leq 1.$ $R \geq 0,$ x
$\{1, \dots, N\}$ $(\sqrt{N}\phi_j(y))_{y,d(y,x)\leq R}$ x
$B_{\mathfrak{X}}(o; R).$
${}^2_j\Phi_{\lambda_j}(d(x; y))$ Φ_{λ_j}
$(q+1)$- $\mathfrak{X}.$
$_j \neq 0,$
$o(\ )$ N^α 11,
(21),
$N),$

.2. **From graph Laplacians to Hecke operators.**

$\Delta_{\mathbb{S}^2}$ $\mathbb{S}^2$

, M 3

$\Delta_{\mathbb{S}^2}$, “ ”

$\mathbb{S}^2$ $g_1, \ldots, g_k$ fi $SO(3)$,

$$T_k f(x) = \sum_{j=1}^{k} \left(f(g_j x) + f(g_j^{-1} x) \right)$$

$\Delta_{\mathbb{S}^2}$.

Theorem 19 (, M 3)**.** *Assume that $g_1, \ldots, g_k$ generate a free subgroup of $SO(3)$.*

For each ℓ, let $(\psi_j^{(\ell)})_{j=1}^{2\ell+1}$ be an orthonormal family of eigenfunctions of $-\Delta_{\mathbb{S}^2}$ of eigenvalue $\ell(\ell+1)$, that are also eigenfunctions of T_k.

Then for any continuous function a on $\mathbb{S}^2$, we have

$$\frac{1}{2\ell+1} \sum_{j=1}^{2\ell+1} \left| \int_M a(x) |\psi_j^{(\ell)}(x)|^2 d\mathrm{Vol}(x) - \int_M a(x) d\mathrm{Vol}(x) \right|^2 \underset{\ell \longrightarrow \infty}{\longrightarrow} 0.$$

T_k ℓ

T_k ℓ^{-1} .

$2k$- $N \sim \ell^2$, $g_1, \ldots, g_k$

, . , 1

11. .

Remark 20. *T_k is not a pseudodifferential operator, so the argument sketched above to show that the basis $(Y_\ell^m)_{\ell \geq 0, |m| \leq \ell}$ could not satisfy quantum ergodicity does not apply here.*

Remark 21. *We note that for very special choices of rotations – rotations that correspond to norm n elements in an order in a quaternion division algebra, the operators T_k are called Hecke operators. It has been conjectured by Böcherer, Sarnak, and Schulze-Pillot that such joint eigenfunctions satisfy the much stronger quantum unique ergodicity property. This conjecture is still open.*

3 . [2000,

, n)

infinit

almost-every ergodic component,

. [3 ,

M,

,

0 ,

one .

[redacted] 11

.3. Quantum ergodicity o[redacted] Rie[redacted]ann su[redacted]faces of [redacted]igh ge[redacted]s. 1[redacted] 12

[redacted] 11 [redacted]

T[redacted]eorem 22 ([redacted] 0). [redacted] (S_N) *be* [redacted] *s*[redacted]*quen*[redacted] *of hyper*[redacted] *ces, wh*[redacted]*ose ge*[redacted]*s (equivale*[redacted]*y, vol*[redacted]*me) g*[redacted]*es to* [redacted]

(EXP) *Assume th*[redacted] *first e*[redacted]*genva*[redacted]*e* [redacted]$_1$ [redacted] *of* $-\Delta$ [redacted] *is* [redacted]*ed a*[redacted] 0 *as* $N \longrightarrow \infty$

(BSH) *Assume the*[redacted] *are f*[redacted] *sho*[redacted] *geod*[redacted]*s; in o*[redacted]*er words* $(S$[redacted]$)$ *co*[redacted] *the B*[redacted]*njamin*[redacted] *Schramm* [redacted]*se to* [redacted] *hy*[redacted]*rbolic* [redacted] *: for* [redacted] > 0

$$\lim_{N \longrightarrow +\infty} \frac{V[redacted]\{x \in [redacted](x) < [redacted]\}}{V[redacted]_N)} = 0$$

wh[redacted]*re* [redacted] $(x$[redacted] *means the* [redacted]*ectiv*[redacted] *rad*[redacted]*s at* x

Fix an [redacted]*terval* $I \subset$ [redacted]$4; +$[redacted].

Let $(\phi_i^{(}$[redacted]$)$ *be an or*[redacted]*norm*[redacted] *basis* [redacted]*f eige*[redacted]*nctions* [redacted] *the La*[redacted]*aci*[redacted] *on* [redacted]

Let $a = $[redacted]$_N : S_N \longrightarrow$ [redacted] *be s*[redacted] *tha*[redacted] $|a(x)|$ [redacted] *for all* [redacted] $\in S_N$. *The*[redacted]

$$\lim_{N \longrightarrow +} [redacted] \frac{1}{[redacted]ol(S[redacted]} \sum_{\lambda_i(N[redacted] I} \left| \int_{S_N} [redacted])|\phi_i^{(N)}([redacted]|^2 d[redacted] - ([redacted] \right|^2 [redacted] 0$$

wh[redacted]*re* $\langle a \rangle$ [redacted] $\frac{1}{Vol(S_N)} \int_{S}$[redacted]$x)dx$

[redacted] $(1/4;$ [redacted]$)$ [redacted] L^2- [redacted]

[redacted] S_N [redacted] ,

[redacted] ([redacted] 22 [redacted]),

[redacted] 2. .1 [redacted] *random*

[redacted]

- [redacted] *random* [redacted] 1 [redacted]

- [redacted] ,

[redacted] $(\phi_i^{(N}$ [redacted]

[redacted] 12 [redacted]

[redacted] 20 – [redacted]

[redacted] 1, [redacted]

[redacted]), [redacted] g — [redacted] ∞.

g — [redacted] $+\infty$ [redacted]

[redacted] 12 .

[redacted]f[redacted]e[redacted]ences

[redacted]e Lu[redacted] A.[redacted] Al[redacted]ul[redacted] Kra[redacted] a[redacted]d Scardic[redacted]io A. And[redacted] localiz[redacted]n on the bethe [redacted]ttice[redacted] Non[redacted]rgo[redacted]cit[redacted] o[redacted]xten[redacted] *Ph[redacted]. Rev. L[redacted].*, 113[redacted]201[redacted]

[redacted]ichae[redacted] Aiz[redacted]n a[redacted]d [redacted]mone[redacted] [redacted]solutely con[redacted]nuous [redacted]pec[redacted] implies[redacted]listic transport [redacted]r qua[redacted]tum [redacted]rticle[redacted] in [redacted] ran[redacted] [redacted]ree gra[redacted]hs. *J. [redacted]ath[redacted]s.*, 53([redacted]5205, 15, 2012.

[redacted]alini [redacted]nan[redacted]ar[redacted]man[redacted] E[redacted]ntropy [redacted]tion of ei[redacted]nfunc[redacted]ons[redacted] *of Ma[redacted]2)*, 168(2):435–[redacted]75, 20[redacted]8.

[redacted]alini [redacted]nan[redacted]ar[redacted]man[redacted] Q[redacted]antu[redacted] o[redacted] regular g[redacted]phs. *[redacted]omm[redacted]th. Ph[redacted]*[redacted]53(2):633–690, [redacted]17.

[redacted]alini [redacted]nan[redacted]ar[redacted]man[redacted] a[redacted] Eti[redacted]sc[redacted] Quantu[redacted] ergo[redacted]city [redacted]arge re[redacted] graphs. *Duke [redacted]ath.* [redacted], 16[redacted](4)[redacted]23[redacted]6[redacted]2015[redacted]

[redacted]alini [redacted]nan[redacted]ar[redacted]man[redacted]an[redacted] Sté[redacted]p[redacted]n[redacted]cher. En[redacted]opy of[redacted]emi[redacted]al mea[redacted]s of the Walsh-[redacted]anti[redacted]d b[redacted]er[redacted]m[redacted] *[redacted]n. H[redacted]ré*[redacted](1):37–7[redacted] 2007.

[redacted]alini [redacted]nan[redacted]ar[redacted]ma[redacted]a[redacted] Sté[redacted]er[redacted]macher. H[redacted]lf-delo[redacted]aliz[redacted] of eig[redacted]ctions for the [redacted]placi[redacted]n o[redacted] an[redacted] Ano[redacted]o[redacted]manif[redacted]s. *[redacted] Fourier (Greno[redacted])*, [redacted]2465–2[redacted] 2007. Festival [redacted]ves C[redacted]in de V[redacted]

[redacted]alini [redacted]nan[redacted]ar[redacted]man[redacted] and Mos[redacted] P[redacted]sson ker[redacted] expa[redacted]sions [redacted] schrödin[redacted]r operators on [redacted]es. *[redacted]ps://arxiv.org[redacted]05[redacted]7*, 2016.

[redacted]alini [redacted]nan[redacted]ar[redacted]man[redacted] and Mostafa[redacted] [redacted]uantum erg[redacted]icity [redacted]from [redacted]tral to s[redacted]tial delocaliza-[redacted]n ar[redacted]v:1[redacted]4.[redacted]6[redacted] *preprint*, 20[redacted]

[[redacted]alini [redacted]nan[redacted]ar[redacted]man and Mostafa[redacted]ri. Q[redacted]ntum er[redacted]dicity [redacted]or th[redacted]nderson model on regular [redacted]aphs. *J. M[redacted]th. Phys.*, 58(9):0919[redacted]0, 201[redacted]

[[redacted]alini [redacted]nan[redacted]ar[redacted]an and Mostafa S[redacted]i. Rec[redacted]t results[redacted]quan[redacted]m erg[redacted]ity on graphs and further investigation[redacted]*pr[redacted]rint, arXiv:1711.01666*, 20[redacted]

[[redacted] Agnes Back[redacted]usz and Balazs Szegedy. On[redacted]e alm[redacted]st eige[redacted]ector[redacted]of random regular graphs arxiv:1607.0[redacted]785. *preprint*, 2016.

[[redacted] Roland Bauerschmidt, Jiaoyang Huang, Ant[redacted] Knowles, and H[redacted]ng-T[redacted] Yau. Bulk eigenvalue statistics for random regular graphs arxiv:1505.06700. 2015.

[14] Roland Bauerschmidt, Jiaoyang Huang, and Horng-Tzer Yau. Local kesten–mckay law for random regular graphs arxiv:1609.09052. 2016.

[15] Roland Bauerschmidt, Antti Knowles, and Horng-Tzer Yau. Local semicircle law for random regular graphs. *Comm. Pure Appl. Math.*, 70(10):1898–1960, 2017.

[16] Itai Benjamini and Oded Schramm. Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.*, 6:no. 23, 13, 2001.

[17] Pierre H. Bérard. On the wave equation on a compact Riemannian manifold without conjugate points. *Math. Z.*, 155(3):249–276, 1977.

[18] G. Berkolaiko, J. P. Keating, and U. Smilansky. Quantum ergodicity for graphs related to interval maps. *Comm. Math. Phys.*, 273(1):137–159, 2007.

[19] G. Berkolaiko, J. P. Keating, and B. Winn. No quantum ergodicity for star graphs. *Comm. Math. Phys.*, 250(2):259–285, 2004.

[20] Michael Berry. Regular and irregular semiclassical wavefunctions. *J. Phys. A*, 10(12):2083–2091, 1977.

[21] Michael Berry and Michael Tabor. Level clustering in the regular spectrum. *Proc. Royal Soc. A*, pages 375–394, 1977.

[22] Matthew Blair and Christopher Sogge. Concerning toponogov's theorem and logarithmic improvement of estimates of eigenfunctions, to appear in j. di . geom. 2015.

[23] Matthew Blair and Christopher Sogge. Logarithmic improvements in l^p bounds for eigenfunctions at the critical exponent in the presence of nonpositive curvature. 2017.

[24] Matthew D. Blair and Christopher D. Sogge. Refined and Microlocal Kakeya–Nikodym Bounds of Eigenfunctions in Higher Dimensions. *Comm. Math. Phys.*, 356(2):501–533, 2017.

[25] Siegfried Böcherer, Peter Sarnak, and Rainer Schulze-Pillot. Arithmetic and equidistribution of measures on the sphere. *Comm. Math. Phys.*, 242(1-2):67–80, 2003.

[26] O. Bohigas, M.-J. Giannoni, and C. Schmit. Characterization of chaotic quantum spectra and universality of level fluctuation laws. *Phys. Rev. Lett.*, 52(1):1–4, 1984.

[27] Oriol Bohigas. Random matrix theories and chaotic dynamics. In *Chaos et physique quantique (Les Houches, 1989)*, pages 87–199. North-Holland, Amsterdam, 1991.

[28] Niels Bohr. On the constitution of atoms and molecules. *Philosophical Magazine*, 26:1–24, 1913.

[29] Jens Bolte and Rainer Glaser. A semiclassical Egorov theorem and quantum ergodicity for matrix valued operators. *Comm. Math. Phys.*, 247(2):391–419, 2004.

[30] Yannick Bonthonneau. The Θ function and the Weyl law on manifolds without conjugate points. *Doc. Math.*, 22:1275–1283, 2017.

[31] Yannick Bonthonneau and Steve Zelditch. Quantum ergodicity for Eisenstein functions. *C. R. Math. Acad. Sci. Paris*, 354(9):907–911, 2016.

[32] David Borthwick. *Spectral theory of infinite-area hyperbolic surfaces*, volume 256 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2007.

[33] P. Bourgade and H.-T. Yau. The eigenvector moment flow and local quantum unique ergodicity. *preprint*, 2013.

[34] Jean Bourgain and Elon Lindenstrauss. Entropy of quantum limits. *Comm. Math. Phys.*, 233(1):153–171, 2003.

[35] Louis Brillouin. La mécanique ondulatoire de schrödinger; une méthode générale de résolution par approximations successives. *C. R. A. S.*, (183):24–26, 1926.

[36] Shimon Brooks and Étienne Le Masson. l^p-norms of eigenfunctions on regular graphs and on the sphere arxiv:1710.10922 d. *preprint*, 2017.

[37] Shimon Brooks, Étienne Le Masson, and Elon Lindenstrauss. Quantum ergodicity and averaging operators on the sphere. *Int. Math. Res. Not.*, 2015. To appear, arXiv:1505.03887.

[38] Shimon Brooks and Elon Lindenstrauss. Non-localization of eigenfunctions on large regular graphs. *Israel J. Math.*, 193(1):1–14, 2013.

[39] Shimon Brooks and Elon Lindenstrauss. Joint quasimodes, positive entropy, and quantum unique ergodicity. *Invent. Math.*, 198(1):219–259, 2014.

[40] Y. Colin de Verdière. Ergodicité et fonctions propres du laplacien. *Comm. Math. Phys.*, 102(3):497–502, 1985.

[41] Y. Colin de Verdière. Semi-classical measures on quantum graphs and the gauss map of the determinant manifold. *Ann. H. Poincaré, to appear*, 2013.

[42] Yves Colin de Verdière, Luc Hillairet, and Emmanuel Trélat. Spectral asymptotics for sub-riemannian laplacians. i: quantum ergodicity and quantum limits in the 3d contact case. *preprint*, 2015.

[43] Dominique Delande. Chaos in atomic and molecular physics. In *Chaos et physique quantique (Les Houches, 1989)*, pages 665–726. North-Holland, Amsterdam, 1991.

[44] Persi Diaconis and Daniel Stroock. Geometric bounds for eigenvalues of Markov chains. *Ann. Appl. Probab.*, 1(1):36–61, 1991.

[45] Ioana Dumitriu and Soumik Pal. Sparse regular random graphs: spectral density and eigenvectors. *Ann. Probab.*, 40(5):2197–2235, 2012.

[46] Semyon Dyatlov and Colin Guillarmou. Microlocal limits of plane waves and Eisenstein functions. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(2):371–448, 2014.

[47] Semyon Dyatlov and Long Jin. Semiclassical measures on hyperbolic surfaces have full support arxiv:1705.05019. *preprint*, 2017.

[48] Manfred Einsiedler and Elon Lindenstrauss. Diagonalizable flows on locally homogeneous spaces and number theory. In *International Congress of Mathematicians. Vol. II*, pages 1731–1759. Eur. Math. Soc., Zürich, 2006.

[49] Albert Einstein. über einen die erzeugung und verwandlung des lichts betre enden heuristischen gesichtspunkt. *Annalen der Physik*, 17, 1905.

[50] Albert Einstein. Zum quantensatz von sommerfeld und epstein. *Verhandl. deut. physik. Ges.*, 1917.

[51] László Erd˝ os, Antti Knowles, Horng-Tzer Yau, and Jun Yin. Spectral statistics of Erdös-Rényi graphs I: Local semicircle law. *Ann. Probab.*, 41(3B):2279–2375, 2013.

[52] László Erdős, Benjamin Schlein, and Horng-Tzer Yau. Local semicircle law and complete delocalization for Wigner random matrices. *Comm. Math. Phys.*, 287(2):641–655, 2009.

[53] László Erdős, Benjamin Schlein, and Horng-Tzer Yau. Semicircle law on short scales and delocalization of eigenvectors for Wigner random matrices. *Ann. Probab.*, 37(3):815–852, 2009.

[54] Frédéric Faure and Stéphane Nonnenmacher. On the maximal scarring for quantum cat map eigenstates. *Comm. Math. Phys.*, 245(1):201–214, 2004.

[55] Frédéric Faure, Stéphane Nonnenmacher, and Stephan De Bièvre. Scarred eigenstates for quantum cat maps of minimal periods. *Comm. Math. Phys.*, 239(3):449–492, 2003.

[56] Leander Geisinger. Convergence of the density of states and delocalization of eigenvectors on random regular graphs. *J. Spectr. Theory*, 5(4):783–827, 2015.

[57] Patrick Gérard and Éric Leichtnam. Ergodic properties of eigenfunctions for the Dirichlet problem. *Duke Math. J.*, 71(2):559–607, 1993.

[58] S. Gnutzmann, J. P. Keating, and F. Piotet. Eigenfunction statistics on quantum graphs. *Ann. Physics*, 325(12):2595–2640, 2010.

[59] Colin Guillarmou and Frédéric Naud. Equidistribution of Eisenstein series on convex co-compact hyperbolic manifolds. *Amer. J. Math.*, 136(2):445–479, 2014.

[60] Xiaolong Han. Small scale quantum ergodicity in negatively curved manifolds. *Nonlinearity*, 28(9):3263–3288, 2015.

[61] Xiaolong Han. Small scale equidistribution of random eigenbases. *Comm. Math. Phys.*, 349(1):425–440, 2017.

[62] Andrew Hassell. Ergodic billiards that are not quantum unique ergodic. *Ann. of Math. (2)*, 171(1):605–619, 2010. With an appendix by the author and Luc Hillairet.

[63] Andrew Hassell and Melissa Tacy. Improvement of eigenfunction estimates on manifolds of nonpositive curvature. *Forum Math.*, 27(3):1435–1451, 2015.

[64] Werner Heisenberg. über quantentheoretische umdeutung kinematischer und mechanischer beziehungen. *Zeitschrift f. Physik*, 33:879–893, 1925.

[65] B. Hel er, A. Martinez, and D. Robert. Ergodicité et limite semi-classique. *Comm. Math. Phys.*, 109(2):313–326, 1987.
[66] Eric Heller. Wavepacket dynamics and quantum chaology. In *Chaos et physique quantique (Les Houches, 1989)*, pages 549–661. North-Holland, Amsterdam, 1991.
[67] Hamid Hezari and Gabriel Rivière. L^p norms, nodal sets, and quantum ergodicity. *Adv. Math.*, 290:938–966, 2016.
[68] Maxime Ingremeau. Distorted plane waves on manifolds of nonpositive curvature. *Comm. Math. Phys.*, 350(2):845–891, 2017.
[69] Dmitry Jakobson. Quantum unique ergodicity for Eisenstein series on $\mathrm{PSL}_2(\mathbf{Z})\backslash\mathrm{PSL}_2(\mathbf{R})$. *Ann. Inst. Fourier (Grenoble)*, 44(5):1477–1504, 1994.
[70] Dmitry Jakobson, Yuri Safarov, and Alexander Strohmaier. The semiclassical theory of discontinuous systems and ray-splitting billiards. *Amer. J. Math.*, 137(4):859–906, 2015. With an appendix by Yves Colin de Verdière.
[71] Dmitry Jakobson and Alexander Strohmaier. High energy limits of Laplace-type and Dirac-type eigenfunctions and frame flows. *Comm. Math. Phys.*, 270(3):813–833, 2007.
[72] Dmitry Jakobson, Alexander Strohmaier, and Steve Zelditch. On the spectrum of geometric operators on Kähler manifolds. *J. Mod. Dyn.*, 2(4):701–718, 2008.
[73] J. P. Keating. Quantum graphs and quantum chaos. In *Analysis on graphs and its applications*, volume 77 of *Proc. Sympos. Pure Math.*, pages 279–290. Amer. Math. Soc., Providence, RI, 2008.
[74] J. P. Keating, J. Marklof, and B. Winn. Value distribution of the eigenfunctions and spectral determinants of quantum star graphs. *Comm. Math. Phys.*, 241(2-3):421–452, 2003.
[75] Harry Kesten. Symmetric random walks on groups. *Trans. Amer. Math. Soc.*, 92:336–354, 1959.
[76] Abel Klein. Extended states in the Anderson model on the Bethe lattice. *Adv. Math.*, 133(1):163–184, 1998.
[77] Tsampikos Kottos and Uzy Smilansky. Quantum chaos on graphs. *Phys. Rev. Lett.*, 79:4794–7, 1997.
[78] Tsampikos Kottos and Uzy Smilansky. Periodic orbit theory and spectral statistics for quantum graphs. *Ann. Physics*, 274(1):76–124, 1999.
[79] Hendrik Anthony Kramers. Wellenmechanik und halbzahlige quantisierung. *Zeitschrift f. Physik*, (39):828–840, 1926.
[80] Étienne Le Masson and Tuomas Sahlsten. Quantum ergodicity and benjamini-schramm convergence of hyperbolic surfaces. arxiv:1605.05720. *Duke Math. J.*, to appear.
[81] F. Ledrappier and L.-S. Young. The metric entropy of di eomorphisms. I. Characterization of measures satisfying Pesin's entropy formula. *Ann. of Math. (2)*, 122(3):509–539, 1985.
[82] F. Ledrappier and L.-S. Young. The metric entropy of di eomorphisms. II. Relations between entropy, exponents and dimension. *Ann. of Math. (2)*, 122(3):540–574, 1985.
[83] Stephen Lester and Zeév Rudnick. Small scale equidistribution of eigenfunctions on the torus. *Comm. Math. Phys.*, 350(1):279–300, 2017.
[84] Elon Lindenstrauss. Invariant measures and arithmetic quantum unique ergodicity. *Ann. of Math. (2)*, 163(1):165–219, 2006.
[85] Elon Lindenstrauss. Equidistribution in homogeneous spaces and number theory. In *Proceedings of the International Congress of Mathematicians. Volume I*, pages 531–557. Hindustan Book Agency, New Delhi, 2010.
[86] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.
[87] W. Luo and P. Sarnak. Number variance for arithmetic hyperbolic surfaces. *Comm. Math. Phys.*, 161(2):419–432, 1994.
[88] Jens Marklof. Pair correlation densities of inhomogeneous quadratic forms. *Ann. of Math. (2)*, 158(2):419–471, 2003.
[89] Brendan D. McKay. The expected eigenvalue distribution of a large regular graph. *Linear Algebra Appl.*, 40:203–216, 1981.

[90] Brendan D. McKay, Nicholas C. Wormald, and Beata Wysocka. Short cycles in random regular graphs. *Electron. J. Combin.*, 11(1):Research Paper 66, 12 pp. (electronic), 2004.
[91] Maryam Mirzakhani. On Weil-Petersson volumes and geometry of random hyperbolic surfaces. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 1126–1145. Hindustan Book Agency, New Delhi, 2010.
[92] Maryam Mirzakhani. Growth of Weil-Petersson volumes and random hyperbolic surfaces of large genus. *J. Differential Geom.*, 94(2):267–300, 2013.
[93] Max Planck. über eine verbesserung der wienschen spektralgleichung. *Verhandl. deut. physik. Ges.*, 13:202–204, 1900.
[94] Gabriel Rivière. Entropy of semiclassical measures in dimension 2. *Duke. Math. J.*, 155(2):271–335, 2010.
[95] Zeév Rudnick and Peter Sarnak. The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Comm. Math. Phys.*, 161(1):195–213, 1994.
[96] Peter Sarnak. Arithmetic quantum chaos. In *The Schur lectures (1992) (Tel Aviv)*, volume 8 of *Israel Math. Conf. Proc.*, pages 183–236. Bar-Ilan Univ., Ramat Gan, 1995.
[97] Peter Sarnak. Values at integers of binary quadratic forms. In *Harmonic analysis and number theory (Montreal, PQ, 1996)*, volume 21 of *CMS Conf. Proc.*, pages 181–203. Amer. Math. Soc., Providence, RI, 1997.
[98] Peter Sarnak. Spectra of hyperbolic surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 40(4):441–478, 2003.
[99] Erwin Schrödinger. Quantisierung als eigenwertproblem (erste mitteilung). *Annalen der Physik (4)*, 79:361–376, 1926.
[100] Erwin Schrödinger. Quantisierung als eigenwertproblem (zweite mitteilung). *Annalen der Physik (4)*, 79:489–527, 1926.
[101] Erwin Schrödinger. über das verhältnis der heisenberg–born–jordanschen quantenmechanik zu der meinen. *Annalen der Physik (4)*, 79:734–756, 1926.
[102] Atle Selberg. Remarks on the distribution of poles of Eisenstein series. In *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989)*, volume 3 of *Israel Math. Conf. Proc.*, pages 251–278. Weizmann, Jerusalem, 1990.
[103] Martin Sieber and Klaus Richter. Correlations between periodic orbits and their rôle in spectral statistics. *Physica Scripta*, T90, 2001.
[104] Uzy Smilansky. Quantum chaos on discrete graphs. *J. Phys. A*, 40(27):F621–F630, 2007.
[105] Uzy Smilansky. Discrete graphs – a paradigm model for quantum chaos. *Séminaire Poincaré*, XIV:1–26, 2010.
[106] A. I. Šnirelʹman. Ergodic properties of eigenfunctions. *Uspehi Mat. Nauk*, 29(6(180)):181–182, 1974.
[107] Christopher D. Sogge. Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds. *J. Funct. Anal.*, 77(1):123–138, 1988.
[108] Christopher D. Sogge and Steve Zelditch. Riemannian manifolds with maximal eigenfunction growth. *Duke Math. J.*, 114(3):387–437, 2002.
[109] Christopher D. Sogge and Steve Zelditch. Focal points and sup-norms of eigenfunctions. *Rev. Mat. Iberoam.*, 32(3):971–994, 2016.
[110] Christopher D. Sogge and Steve Zelditch. Focal points and sup-norms of eigenfunctions II: the two-dimensional case. *Rev. Mat. Iberoam.*, 32(3):995–999, 2016.
[111] K. Soundararajan. Quantum unique ergodicity and number theory. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 357–382. Hindustan Book Agency, New Delhi, 2010.
[112] Kannan Soundararajan. Quantum unique ergodicity for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. *Ann. of Math. (2)*, 172(2):1529–1538, 2010.
[113] Linh V. Tran, Van H. Vu, and Ke Wang. Sparse random graphs: eigenvalues and eigenvectors. *Random Structures Algorithms*, 42(1):110–134, 2013.
[114] Je rey M. VanderKam. L^∞ norms and quantum ergodicity on the sphere. *Internat. Math. Res. Notices*, (7):329–347, 1997.

[115] Gregor Wentzel. Eine verallgemeinerung der quantenbedingungen für die zwecke der wellenmechanik. *Zeitschrift f. Physik*, (38):518–529, 1926.
[116] Steven Zelditch. Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke Math. J.*, 55(4):919–941, 1987.
[117] Steven Zelditch. Mean Lindelöf hypothesis and equidistribution of cusp forms and Eisenstein series. *J. Funct. Anal.*, 97(1):1–49, 1991.
[118] Steven Zelditch. Quantum ergodicity on the sphere. *Comm. Math. Phys.*, 146(1):61–71, 1992.
[119] Steven Zelditch and Maciej Zworski. Ergodicity of eigenfunctions for ergodic billiards. *Comm. Math. Phys.*, 175(3):673–682, 1996.

IRMA, 7 rue René-Descartes, 67084 STRASBOURG CEDEX, FRANCE
E-mail address: anantharaman@unistra.fr