

# Backward Stochastic Volterra Integral Equations

Jiongmin Yong

University of Central Florida

July, 2010

# Outline

1. Introduction — Motivations
2. Definition of Solution
3. Well-Posedness of BSVIEs
4. Properties of Solutions
5. Some Remarks

# 1. Introduction — Motivations

$(\Omega, \mathcal{F}, \mathbb{F}, P)$  | a complete filtered probability space

$W(\cdot)$  | a one-dimensional standard Brownian motion

$\mathbb{F} \equiv \{\mathcal{F}_t\}_{t \geq 0}$  | natural filtration of  $W(\cdot)$ , augmented by all  $P$ -null sets

Consider FSDE:

$$(1.1) \quad \begin{aligned} & \begin{cases} \dot{X}(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \\ X(0) = x. \end{cases} \end{aligned}$$

Equivalent to:

$$(1.2) \quad X(t) = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s).$$

General forward stochastic Volterra integral equation: (FSVIE)

$$(1.3) \quad X(t) = \varphi(t) + \int_0^t b(t, s, X(s))ds + \int_0^t \sigma(t, s, X(s))dW(s).$$

- In general, FSVIE (1.3) cannot be transformed into a form of FSDE (1.1).
- FSVIE (1.3) allows some long-range dependence on the noises.
- Could allow  $\sigma(t, s, X(s))$  to be  $\mathcal{F}_t$ -measurable, still might have adapted solutions (Pardoux–Protter, 1990).
- May model wealth process involving investment delay, etc. (Duffie–Huang, 1986).

Consider BSDE:

$$(1.4) \quad \begin{cases} dY(t) = -g(t, Y(t), Z(t))dt + Z(t)dW(t), & t \in [0, T], \\ Y(T) = \xi. \end{cases}$$

- Linear case was introduced by Bismut (1973).
- Nonlinear case was introduced by Pardoux–Peng (1990).
- Can be applied to (European) contingent claim pricing, stochastic differential utility, dynamic risk measures, ...
- Leads to nonlinear Feynman-Kac formula, pointwise convergence in homogenization problems, nonlinear expectation, ...

BSDE (1.4) is equivalent to

$$(1.5) \quad Y(t) = \xi + \int_t^T g(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s).$$

Called a backward stochastic Volterra integral equation (BSVIE).

**Recall:**

$$(1.2) \quad X(t) = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s).$$

$$(1.3) \quad X(t) = \varphi(t) + \int_0^t b(t, s, X(s))ds + \int_0^t \sigma(t, s, X(s))dW(s).$$

**Question:**

What is the analog of (1.3) for (1.5) as (1.3) for (1.2)?

## A Proposed Form:

$$(1.6) \quad Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_t^T Z(t, s) dW(s), \quad t \in [0, T],$$

$(Y(\cdot), Z(\cdot, \cdot))$  | unknown process

### Remarks:

- The term  $Z(t, s)$  depends on  $t$  and  $s$ ;
- The drift depends on both  $Z(t, s)$  and  $Z(s, t)$ .
- (1.6) is strictly more general than BSDE (1.5).
- $\psi(\cdot)$  does not have to be  $F$ -adapted.
- Need  $Z(t, \cdot)$  to be  $F$ -adapted, and  $\int_0^T |Z(t, s)|^2 ds < \infty$ , a.e.  $t \in [0, T]$ , a.s.

By taking conditional expectation on (1.6), we have

$$Y(t) = E \left[ \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t)) ds \mid \mathcal{F}_t \right].$$

This leads to the **second** interesting motivation.

- Expected discounted utility (process) has the form:

$$Y(t) = E \left[ \xi e^{-\beta(T-t)} + \int_t^T u(C(s)) e^{-\beta(s-t)} ds \mid \mathcal{F}_t \right], \quad t \in [0, T].$$

$C(\cdot)$  — consumption process,  $u(\cdot)$  — utility function

$\beta$  — discount rate,  $\xi$  — terminal time wealth

- Expected discounted utility is equivalent to a linear BSDE:

$$Y(t) = \xi + \int_t^T \left( -\beta Y(s) + C(u(s)) \right) ds - \int_t^T Z(s) dW(s).$$



- $e^{-\beta(s-t)}$  exhibits a time-consistent memory effect. If the memory is not time-consistent, the utility process will not be a solution to a BSDE! But, it might be a solution to a BSVIE!
- Duffie–Epstein (1992) introduced stochastic differential utility:

$$Y(t) = E \left[ \xi + \int_t^T g(s, Y(s)) ds \mid \mathcal{F}_t \right], \quad t \in [0, T].$$

which is equivalent to a nonlinear BSDE:

$$Y(t) = \xi + \int_t^T g(s, Y(s)) ds - \int_t^T Z(s) dW(s).$$



## 2. Definition of Solutions.

Let  $H = \mathbb{R}^m, \mathbb{R}^{m \times d}$ , etc., with norm  $|\cdot|_a$

$$L^2(\Omega) = \xi : \Omega \rightarrow H \quad \xi \in \mathcal{F}_T, E|\xi|^2 < \infty,$$

$$L^2((0, T) \times \Omega) = \varphi : (0, T) \times \Omega \rightarrow H$$

$\varphi$  is  $\mathcal{B}([0, T]) \otimes \mathcal{F}_T$ -measurable,  $E \int_0^T |\varphi(t)|^2 dt < \infty$ ,

$$L^2_{\mathbb{F}}(0, T) = \varphi \in L^2((0, T) \times \Omega), \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted}.$$

$$L^2(0, T; L^2_{\mathbb{F}}(0, T)) = Z; [0, T]^2 \times \Omega \rightarrow H$$

$Z(t, \cdot)$  is  $\mathbb{F}$ -adapted, a.e.  $t \in [0, T]$ ,

$$E \int_0^T \int_0^T |Z(t, s)|^2 ds dt < \infty.$$

Recall:

$$(2.1) \quad Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_t^T Z(t, s) dW(s), \quad t \in [0, T],$$

Similar to BSDEs, it seems to be reasonable to introduce

**Definition 2.1.**  $(Y, Z) \in L^2_{\mathbb{F}}(0, T) \times L^2(0, T; L^2_{\mathbb{F}}(0, T))$  satisfying (2.1) is called an *adapted solution* of BSVIE (2.1).

**Example 2.2.** Consider BSVIE:

$$(2.2) \quad Y(t) = \int_t^T Z(s, t) ds - \int_t^T Z(t, s) dW(s), \quad t \in [0, T].$$

We can check that

$$\begin{aligned} \begin{matrix} \exists \\ < \\ \vdots \end{matrix} & \quad Y(t) = (T - t)\zeta(t), \quad t \in [0, T], \\ & \quad Z(t, s) = I_{[0, t]}(s)\zeta(s), \quad (t, s) \in [0, T] \times [0, T], \end{aligned}$$

is an adapted solution of (2.2) for any  $\zeta(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R})$ . Thus, adapted solutions are not unique!

## Observation:

$$(2.1) \quad Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_t^T Z(t, s) dW(s), \quad t \in [0, T],$$

does not give enough "restrictions" on  $Z(t, s)$  with  $0 \leq s \leq t \leq T$ .

Need to "specify"  $Z(t, s)$  for  $0 \leq s \leq t \leq T$ .

**Definition 2.3.**  $(Y, Z) \in L^2_{\mathbb{F}}(0, T) \times L^2(0, T; L^2_{\mathbb{F}}(0, T))$  is called an *adapted M-solution* of (2.1) if (2.1) is satisfied and also

$$(2.3) \quad Y(t) = \mathbb{E}Y(t) + \int_0^t Z(t, s) dW(s), \quad t \in [0, T].$$

### 3. Well-posedness of BSVIEs.

(H1) Map  $g$  is measurable satisfying

$$\mathbb{E} \int_0^T \int_t^T |g(t, s, 0, 0)|^2 ds dt < \infty,$$

and exists a (deterministic) function  $L$  with

$$\sup_{t \in [0, T]} \int_t^T L(t, s)^{2+\varepsilon} ds < \infty,$$

for some  $\varepsilon > 0$  such that

$$\begin{aligned} & |g(t, s, y, z, \zeta) - g(t, s, \bar{y}, \bar{z}, \bar{\zeta})| \\ & \leq L(t, s) (|y - \bar{y}| + |z - \bar{z}| + |\zeta - \bar{\zeta}|). \end{aligned}$$

**Theorem 3.1.** Let (H1) hold. Then  $\forall \psi$ , (2.1) admits a unique adapted M-solution  $(Y, Z)$ . Moreover: for any  $r \in [0, T]$ ,

$$(3.1) \quad \begin{aligned} & \int_r^T \mathbb{E} |Y(t)|^2 dt + \int_r^T \int_r^T \mathbb{E} |Z(t, s)|^2 ds dt \\ & \leq C \int_r^T \mathbb{E} |\psi(t)|^2 dt + \int_r^T \int_r^T |g(t, s, 0, 0)| ds dt. \end{aligned}$$

If  $(\bar{Y}, \bar{Z})$  is the adapted M-solution corresponding to  $\bar{\psi}$ , then

$$(3.2) \quad \begin{aligned} & \int_r^T \mathbb{E} |Y(t) - \bar{Y}(t)|^2 dt + \int_r^T \int_r^T \mathbb{E} |Z(t, s) - \bar{Z}(t, s)|^2 ds dt \\ & \leq C \int_r^T \mathbb{E} |\psi(t) - \bar{\psi}(t)|^2 dt, \quad \forall r \in [0, T]. \end{aligned}$$



## A Difference between BSDEs and BSVIEs:

For BSDE

$$\begin{aligned} Y(t) &= \xi + \int_t^T g(s, Y(s), Z(s)) ds - \int_t^T Z(s) dW(s) \\ &= \xi + \int_{T-\delta}^t g(s, Y(s), Z(s)) ds - \int_{T-\delta}^t Z(s) dW(s) \\ &\quad + \int_t^{T-\delta} g(s, Y(s), Z(s)) ds - \int_t^{T-\delta} Z(s) dW(s) \\ &= Y(T-\delta) + \int_t^{T-\delta} g(s, Y(s), Z(s)) ds - \int_t^{T-\delta} Z(s) dW(s), \\ &\quad t \in [0, T-\delta]. \end{aligned}$$

Thus, one can obtain the solvability on  $[T-\delta, T]$ , then on  $[T-2\delta, T-\delta]$ , etc., to get solvability on  $[0, T]$ .

For BSVIE: (with  $t \in [0, T - \delta]$ )

$$\begin{aligned}
 Y(t) &= \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_t^T Z(t, s) dW(s) \\
 &= \psi(t) + \int_{T-\delta}^T g(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_{T-\delta}^T Z(t, s) dW(s) \\
 &\quad + \int_t^{T-\delta} g(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_t^{T-\delta} Z(t, s) dW(s) \\
 &\equiv \phi(t) + \int_t^{T-\delta} g(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_t^{T-\delta} Z(t, s) dW(s),
 \end{aligned}$$

where it is not obvious if  $\phi(t)$  is/can be chosen  $\mathcal{F}_{T-\delta}$ -measurable!

## 4. Properties of Solutions.

- *A Duality Principle*

ODE case: Consider

$$(4.1) \quad \dot{x}(t) = Ax(t) + f(t), \quad x(0) = 0,$$

$$(4.2) \quad \dot{y}(t) = -A^T y(t) - g(t), \quad y(T) = 0.$$

Then

$$\frac{d}{dt} \langle x(t), y(t) \rangle = \langle f(t), y(t) \rangle - \langle x(t), g(t) \rangle.$$

Thus,

$$(4.3) \quad \int_0^T \langle x(t), g(t) \rangle dt = \int_0^T \langle y(t), f(t) \rangle dt.$$

- (4.2) is called an adjoint equation of (4.1).
- (4.3) is called a duality between (4.1) and (4.2).
- (linear) SDE and BSDE have a similar duality principle. Itô's formula is commonly used.

**Theorem 4.1.** Let  $\varphi \in L^2_{\mathbb{F}}(0, T)$  and  $\psi \in L^2_Z((0, T) \times \Omega)$ . Let

$$(4.4) \quad X(t) = \varphi(t) + \int_0^t A_0(t, s)X(s)ds + \int_0^t A_1(t, s)X(s)dW(s),$$

$$(4.5) \quad \begin{aligned} Y(t) = & \psi(t) + \int_t^T A_0(s, t)^T Y(s) + A_1(s, t)^T Z(s, t) ds \\ & - \int_t^T Z(t, s)dW(s), \quad t \in [0, T]. \end{aligned}$$

Then the following relation holds:

$$(4.6) \quad \mathbb{E} \int_0^T \langle Y(t), \varphi(t) \rangle dt = \mathbb{E} \int_0^T \langle \psi(t), X(t) \rangle dt.$$

(4.5) | the adjoint equation of (4.4)

(4.6) | the duality between (4.4) and (4.5).

- *A Comparison Theorem*

Consider BSDEs: ( $k = 1, 2$ )

$$(4.7) \quad \begin{aligned} &< dY^k(t) = -g^k(t, Y^k(t), Z^k(t))dt + Z^k(t)dW(t), \\ &: Y^k(T) = \xi^k. \end{aligned}$$

Let

$$(4.8) \quad \begin{aligned} &< g^1(t, s, y, z) \leq g^2(t, s, y, z), \quad \forall(t, s, y, z), \\ &: \xi^1 \leq \xi^2, \quad \text{a.s.} \end{aligned}$$

Then

$$(4.9) \quad Y^1(t) \leq Y^2(t), \quad t \in [0, T], \text{ a.s.}$$

- Itô formula is used in the proof.
- Does not rely on the comparison of FSDEs.

**Theorem 4.2.** For  $k = 1, 2$ , let  $g^k : [0, T]^2 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\psi^k(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R})$  such that

$$(4.10) \quad \begin{aligned} &< g^1(t, s, y, \zeta) \leq g^2(t, s, y, \zeta), \quad \forall (t, s, y, \zeta), \\ &: \psi^1(t) \leq \psi^2(t), \quad t \in [0, T], \text{ a.s.} \end{aligned}$$

Let  $(Y^k(\cdot), Z^k(\cdot, \cdot))$  be the adapted M-solution of BSIVE

$$(4.11) \quad \begin{aligned} Y^k(t) = \psi^k(t) + & \int_t^T g^k(t, s, Y^k(s), Z^k(s, t)) ds \\ & - \int_t^T Z^k(t, s) dW(s). \end{aligned}$$

Then the following holds:

$$(4.12) \quad Y^1(t) \leq Y^2(t), \quad \forall t \in [0, T].$$

- *Sub-Additivity and Convexity.*

Let  $(Y(\cdot), Z(\cdot, \cdot))$  be the adapted solution of BSVIE

$$(4.13) \quad Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(s, t)) ds - \int_t^T Z(t, s) dW(s).$$

Denote

$$(4.14) \quad \rho(t; \psi(\cdot)) = Y(t), \quad t \in [0, T].$$

- $\psi(\cdot) \mapsto \rho(t; -\psi(\cdot))$  is essentially a *dynamic risk measure*.

**Proposition 4.4.** Let  $g : [0, T]^2 \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ .

(i) Suppose  $(y, \zeta) \mapsto g(t, s, y, \zeta)$  is sub-additive:

$$g(t, s, y_1 + y_2, \zeta_1 + \zeta_2) \leq g(t, s, y_1, \zeta_1) + g(t, s, y_2, \zeta_2),$$
$$\forall (t, s) \in [0, T]^2, y_1, y_2 \in \mathbb{R}, \zeta_1, \zeta_2 \in \mathbb{R}^d, \text{ a.s. ,}$$

Then  $\psi(\cdot) \mapsto \rho(t; \psi(\cdot))$  is sub-additive:

$$\rho(t; \psi_1(\cdot) + \psi_2(\cdot)) \leq \rho(t; \psi_1(\cdot)) + \rho(t; \psi_2(\cdot)), \quad t \in [0, T], \text{ a.s.}$$



(ii) Suppose  $(y, z) \mapsto g(t, s, y, \zeta)$  is convex:

$$\begin{aligned} &g(t, s, \lambda y_1 + (1 - \lambda)y_2, \lambda \zeta_1 + (1 - \lambda)\zeta_2) \\ &\leq \lambda g(t, s, y_1, \zeta_1) + (1 - \lambda)g(t, s, y_2, \zeta_2), \\ &\quad \forall (t, s) \in [0, T]^2, y_1, y_2 \in \mathbb{R}, \zeta_1, \zeta_2 \in \mathbb{R}^d, \text{ a.s.}, \lambda \in [0, 1]. \end{aligned}$$

Then  $\psi(\cdot) \mapsto \rho(t; \psi(\cdot))$  is convex:

$$\begin{aligned} \rho(t; \lambda \psi_1(\cdot) + (1 - \lambda)\psi_2(\cdot)) &\leq \lambda \rho(t; \psi_1(\cdot)) + (1 - \lambda)\rho(t; \psi_2(\cdot)), \\ t \in [0, T], \text{ a.s.}, \lambda &\in [0, 1]. \end{aligned}$$

- Similar results hold if exchanging super-additivity and sub-additivity, convexity and concavity, respectively.

## 5. Some Remarks:

- Regularity of adapted M-solutions:

$$(1.6) \quad Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_t^T Z(t, s) dW(s), \quad t \in [0, T].$$

Continuity of  $t \mapsto Y(t)$  is not trivial. Malliavin calculus will be involved.

- Necessary conditions for optimal control of FSVIEs can be obtained.
- Existence of dynamic risk measure for general position processes.

.....

**Thank You!**