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# Gradient estimates of Poisson equations on a Riemannian manifold and applications

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# 1. Model and Questions Let

•  $\mu = e^{-V(x)} dx/Z$  (*Z* is the normalization constant) with  $V \in C^1$  on a complete connected Riemannian manifold *M* with convex or empty boundary  $\partial M$ 

• the diffusion  $(X_t)$  generated by  $\mathcal{L} = \Delta - \nabla V \cdot \nabla (\Delta, \nabla \text{ are respectively})$ the Laplacian and the gradient on M is  $\mu$ -reversible and the corresponding Dirichlet form is given by

$$\mathcal{E}_{\mu}(g,g) = \int_{M} |
abla g|^2 \, d\mu, \ g \in \mathbb{D}(\mathcal{E}_{\mu}) = H^1(\mathcal{X},\mu)$$

If  $u = f\mu$  with  $0 < f \in C^1(M)$ , then

$$I(\nu|\mu) = \int_{M} |\nabla \sqrt{f}|^2 \, d\mu = \frac{1}{4} \int_{M} \frac{|\nabla f|^2}{f} \, d\mu \tag{1}$$

is the Fisher-Donsker-Varadhan's information.

Question 1. Given g with  $\mu(g) = 0$  (the heat source), the equilibrium heat distribution G satisfies the Poisson equation

$$-\mathcal{L}G = g.$$

How to estimate

$$\|G\|_{Lip} = \sup_{x \in M} |\nabla G|?$$

Question 2. To estimate the best constant  $c_{P,1}$  in the  $L^1$ -Poincaré inequality

$$\int |f-\mu(f)| d\mu \leq c_{P,1} \int |
abla f| d\mu?$$

It is equivalent to (by Bobkov-Houdré (MAMS 97))

$$2\mu(A)[1-\mu(A)] \leq c_{P,1}\mu_\partial(\partial A), \ orall A \subset M.$$

which is a variant of Cheeger's isoperimetric inequality.

Question 3. The Gaussian concentration constant  $c_G$ : the best constant such that for all g with  $||g||_{Lip} \leq 1$ ,

$$\mathbb{P}_eta\left(rac{1}{t}\int_0^t g(X_s)\,ds\geq \mu(g)+r
ight)\leq \left\|rac{deta}{d\mu}
ight\|_2\exp\left(-trac{r^2}{2c_G}
ight),\,orall t,r>0.$$

#### 2. Gradient estimate : coupling method

Assume that the Ricci curvature is bounded from below by a constant  $K \in \mathbb{R}$ , i.e.  $Ric_x(X, X) \geq K|X|^2$  for all  $x \in M, X \in T_xM$  (the tangent space at x). Define for every real number  $r \in [0, D]$ ,

 $a(r) = \inf \{ \nabla_x \rho(x, y) \cdot \nabla V(x) + \nabla_y \rho(x, y) \cdot \nabla V(y); \ \rho(x, y) = r, y \notin cut (x, y) \}$ 

where cut(x) is the cut-locus of x. Typically

$$a(r) \geq r \inf_{X \in TM, |X|=1} HessV(X,X), r > 0.$$

Let  $b(r):(-D/2,D/2)
ightarrow\mathbb{R}$  be an odd function such that for  $r\in[0,D/2)$ 

$$b(r) = \begin{cases} -\sqrt{K(n-1)} \tan\left[r\sqrt{\frac{K}{n-1}}\right] - \frac{1}{2}a(2r), & \text{if } K \ge 0\\ \sqrt{K^{-}(n-1)} \tanh\left[r\sqrt{\frac{K^{-}}{n-1}}\right] - \frac{1}{2}a(2r), & \text{if } K < 0. \end{cases}$$
(3)

Consider the one-dimensional generator on (-D/2, D/2) with the Neumann boundary condition :

$$\mathcal{L}^{CW} = rac{d^2}{dr^2} + b(r)rac{d}{dr}.$$

**Theorem 1** (Chen-Wang 97) For the best Poincaré constant  $c_P(\mu)$  in $\int |f - \mu(f)|^2 d\mu \le c_P(\mu) \int |
abla f|^2 d\mu$ 

it holds that

$$c_P(\mu) \leq c_P(\mathcal{L}^{CW}).$$

Method : coupling.

Assume now  $Ric + HessV \geq \tilde{K}$ . Consider

$$ilde{b}(r) = egin{cases} -\sqrt{ ilde{K}(n-1)} ext{tan} \left[ r \sqrt{ ilde{rac{ ilde{K}}{n-1}}} 
ight], & ext{if } ilde{K} \geq 0 \ \sqrt{ ilde{K}^-(n-1)} ext{tanh} \left[ r \sqrt{ ilde{rac{ ilde{K}^-}{n-1}}} 
ight], & ext{if } ilde{K} < 0. \end{cases}$$

Theorem 2 (Bakry-Qian 00) It holds that

 $c_P(\mu) \leq c_P(\mathcal{L}^{BQ})$ 

where

$$\mathcal{L}^{BQ} = rac{d^2}{dr^2} + ilde{b}(r)rac{d}{dr}.$$

Method: gradient estimate of the eigenfunction.

**Theorem 3** (Wu 09 JFA) Let  $\mathcal{L} = \Delta - \nabla V \cdot \nabla$  be defined on a connected complete Riemannian manifold M with empty or  $C^{\infty}$  smooth convex boundary  $\partial M$ , with the Neumann boundary condition, where  $V \in C^2$  such that  $\int_M \rho(x, o)^2 e^{-V(x)} dx < +\infty$ . Assume that there is a sequence of convex relatively compact open domains  $M_n$  increasing to M such that  $\partial M_n$  is  $C^{\infty}$ -smooth and convex, if M is non-compact.

(a) Assume that

$$c_{Lip} := \int_0^{D/2} r \exp\left(\int_0^r b(s) ds\right) dr < +\infty.$$
(4)

Then the Poisson operator  $(-\mathcal{L})^{-1}$  is bounded on  $C_{Lip,0}(M)$  and its norm satisfies

$$\|(-\mathcal{L})^{-1}\|_{Lip} \le c_{Lip} \le \|(-\mathcal{L}^{CW})^{-1}\|_{Lip}$$
(5)

or equivalently if  $-\mathcal{L}G = g$ , then  $||G||_{Lip} \leq c_{Lip} ||g||_{Lip}$ .

(b) For any bounded measurable function g with  $\mu(g) = 0$ , let G be a  $C^1$  solution (in the distribution sense) of the Poisson equation  $-\mathcal{L}G = g$ . Then

$$\sup_{x \in M} |\nabla G|(x) \le c_{\delta} \delta(g),$$

$$c_{\delta} := \frac{1}{2} \int_{0}^{D/2} \exp\left(\int_{0}^{r} b(v) dv\right) dr = \frac{1}{4} m[-D/2, D/2]$$
(6)

where 
$$\delta(g) = \sup_{x 
eq y} |g(x) - g(y)|.$$

**Remarks 1** There is a very rich theory about gradient estimates on Riemannian manifolds, often called Li-Yau's gradient estimates, see the book of Schoen-Yau for account of art and bibliographies.

### 3. Question 2.

**Theorem 4** (Wu JFA09) For the best constant  $c_{P,1}$  in the L<sup>1</sup>-Poincaré inequality,

$$c_{P,1}(\mu) \leq 2c_\delta \leq c_{P,1}(\mathcal{L}^{CW}).$$

References : Buser, Yau, Ledoux etc.

# 4. Question 3.

**Theorem 5 (Guillin-Léonard-Wu-Yao PTRF09)** Let  $c_G > 0$  and let  $(X_t)$  be a  $\mu$ -reversible and ergodic Markov process associated with  $(\mathcal{E}, \mathbb{D}(\mathcal{E}))$ . The statements below are equivalent:

(i) The following transportation-information  $W_1I(c)$  inequality holds true:

$$W_1^2(
u,\mu) \leq 2c_G I(
u|\mu), \ orall 
u; \qquad (W_1I(c))$$

(ii) For all Lipschitz function g on M, r > 0 and  $\beta \in M_1(M)$  such that  $d\beta/d\mu \in L^2(\mu)$ ,

$$\mathbb{P}_eta\left(rac{1}{t}\int_0^t u(X_s)\,ds\geq \mu(u)+r
ight)\leq \left\|rac{deta}{d\mu}
ight\|_2\exp\left(-rac{tr^2}{2c_G\|g\|_{ ext{Lip}}^2}
ight).$$

**Theorem 6**  $W_1I(c_G)$  holds with

 $c_G \leq 2c_{Lip}^2$ .

Open Question. Whether  $c_{LS}(\mu) \leq c_{LS}(\mathcal{L}^{CW})$  ?

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#### **Remarks 2**

- 1.  $d_{\rho_a}$  is the metric associated with the carré-du-champ operator of the diffusion.
- 2. The quantity  $C(\rho)$  in (??) is not innocent: Chen-Wang's variational formula for the spectral gap tells us that :

 $C_P(\mu) = \inf_{
ho} C(
ho, 
ho).$ 

