

ℓ_1 -Regularized Linear Regression: Persistence and Oracle Inequalities

Peter Bartlett
EECS and Statistics
UC Berkeley

slides at <http://www.stat.berkeley.edu/~bartlett>

Joint work with Shahar Mendelson and Joe Neeman.

ℓ_1 -regularized linear regression

- ▶ Random pair: $(X, Y) \sim P$, in $\mathbb{R}^d \times \mathbb{R}$.
- ▶ n independent samples drawn from P :
 $(X_1, Y_1), \dots, (X_n, Y_n)$.
- ▶ Find β so linear function $\langle X, \beta \rangle$ has small risk,

$$P\ell_\beta = P(\langle X, \beta \rangle - Y)^2.$$

Here, $\ell_\beta(X, Y) = (\langle X, \beta \rangle - Y)^2$ is the quadratic loss of the linear prediction.

ℓ_1 -regularized linear regression

- ▶ Random pair: $(X, Y) \sim P$, in $\mathbb{R}^d \times \mathbb{R}$.
- ▶ n independent samples drawn from P :
 $(X_1, Y_1), \dots, (X_n, Y_n)$.
- ▶ Find β so linear function $\langle X, \beta \rangle$ has small risk,

$$P\ell_\beta = P(\langle X, \beta \rangle - Y)^2.$$

Example. ℓ_1 -regularized least squares:

$$\hat{\beta} = \arg \min_{\beta \in \mathbb{R}^d} P_n \ell_\beta + \rho_n \|\beta\|_{\ell_1^d},$$

$$\text{where } P_n \ell_\beta = \frac{1}{n} \sum_{i=1}^n (\langle X_i, \beta \rangle - Y_i)^2, \text{ and } \|\beta\|_{\ell_1^d} = \sum_{j=1}^d |\beta_j|.$$

ℓ_1 -regularized linear regression

Example. ℓ_1 -regularized least squares:

$$\hat{\beta} = \arg \min_{\beta \in \mathbb{R}^d} P_n \ell_{\beta} + \rho_n \|\beta\|_{\ell_1^d} \quad ,$$

$$\text{where } P_n \ell_{\beta} = \frac{1}{n} \sum_{i=1}^n (\langle X_i, \beta \rangle - Y_i)^2, \text{ and } \|\beta\|_{\ell_1^d} = \sum_{j=1}^d |\beta_j|.$$

- ▶ Tends to select **sparse** solutions (few non-zero components β_j).
- ▶ Useful, for example, if $d \gg n$.

l_1 -regularized linear regression

Example. l_1 -regularized least squares:

$$\hat{\beta} = \arg \min_{\beta \in \mathbb{R}^d} P_n \ell_{\beta} + \rho_n \|\beta\|_{\ell_1^d} ,$$

Example. l_1 -constrained least squares:

$$\hat{\beta} = \arg \min_{\|\beta\|_{\ell_1^d} \leq b_n} P_n \ell_{\beta} .$$

[Recall: $\ell_{\beta}(X, Y) = (\langle X, \beta \rangle - Y)^2$.]

ℓ_1 -regularized linear regression

Example. ℓ_1 -regularized least squares:

$$\hat{\beta} = \arg \min_{\beta \in \mathbb{R}^d} P_n \ell_{\beta} + \rho_n \|\beta\|_{\ell_1^d} ,$$

Example. ℓ_1 -constrained least squares:

$$\hat{\beta} = \arg \min_{\|\beta\|_{\ell_1^d} \leq b_n} P_n \ell_{\beta} .$$

Some questions:

- ▶ **Prediction:** Does $\hat{\beta}$ give accurate forecasts?
e.g., How does $P \ell_{\hat{\beta}}$ compare with $P \ell_{\beta^*}$?

$$\text{Here, } \beta^* = \arg \min_{\|\beta\|_{\ell_1^d} \leq b_n} P_n \ell_{\beta} .$$

ℓ_1 -regularized linear regression

Example. ℓ_1 -regularized least squares:

$$\hat{\beta} = \arg \min_{\beta \in \mathbb{R}^d} P_n \ell_{\beta} + \rho_n \|\beta\|_{\ell_1^d} ,$$

Example. ℓ_1 -constrained least squares:

$$\hat{\beta} = \arg \min_{\|\beta\|_{\ell_1^d} \leq b_n} P_n \ell_{\beta} .$$

Some questions:

- ▶ Does $\hat{\beta}$ give accurate forecasts?
e.g., $P \ell_{\hat{\beta}}$ versus $P \ell_{\beta^*} = \min_{\|\beta\|_{\ell_1^d} \leq b_n} P \ell_{\beta}$?
- ▶ **Estimation:** Under assumptions on P , is $\hat{\beta} \approx \beta^*$ correct?
- ▶ **Sparsity Pattern Estimation:** Under assumptions on P , are the non-zeros of $\hat{\beta}$ correct?

Outline of Talk

1. For ℓ_1 -constrained least squares, bounds on $P\ell_{\hat{\beta}} - P\ell_{\beta^*}$.
 - ▶ **Persistence:** (Greenshtein and Ritov, 2004)
For what $d_n, b_n \rightarrow \infty$ does $P\ell_{\hat{\beta}} - P\ell_{\beta^*} \rightarrow 0$?
 - ▶ **Convex Aggregation:** (Tsybakov, 2003)
For $b = 1$ (convex combinations of dictionary functions),
what is rate of $P\ell_{\hat{\beta}} - P\ell_{\beta^*}$?
2. For ℓ_1 -regularized least squares, oracle inequalities.
3. Proof ideas.

ℓ_1 -regularized linear regression

Key Issue: ℓ_β is unbounded, so some key tools (e.g., concentration inequalities) cannot immediately be applied.

- ▶ For (X, Y) bounded, ℓ_β can be bounded using $\|\beta\|_{\ell_1^d}$, but this gives loose prediction bounds.
- ▶ We use chaining to show that metric structures of ℓ_1 -constrained linear functions under P_n and P are similar.

Main Results: Excess Risk

For ℓ_1 -constrained least squares,

$$\hat{\beta} = \arg \min_{\|\beta\|_{\ell_1} \leq b} P_n \ell_{\beta},$$

if X and Y have suitable tail behaviour then, with probability $1 - \delta$,

$$P \ell_{\hat{\beta}} - P \ell_{\beta^*} \leq \frac{c \log^{\alpha}(nd)}{\delta^2} \min \left\{ \frac{b^2}{n} + \frac{d}{n}, \frac{b}{\sqrt{n}} \right\} 1 + \frac{b}{\sqrt{n}} .$$

- ▶ Small d regime: d/n .
- ▶ Large d regime: b/\sqrt{n} .

Main Results: Excess Risk

For ℓ_1 -constrained least squares, with probability $1 - \delta$,

$$Pl_{\hat{\beta}} - Pl_{\beta^*} \leq \frac{c \log^\alpha(nd)}{\delta^2} \min \left\{ \frac{b^2}{n} + \frac{d}{n}, \frac{b}{\sqrt{n}} \right\} \left(1 + \frac{b}{\sqrt{n}} \right).$$

Conditions:

1. PY^2 is bounded by a constant.
2.
 - ▶ $\|X\|_\infty$ bounded a.s.,
 - ▶ X log concave and $\max_j \|\langle X, e_j \rangle\|_{L_2} \leq c$, or
 - ▶ X log concave and isotropic.

Application: Persistence

Consider ℓ_1 -constrained least squares,

$$\hat{\beta} = \arg \min_{\|\beta\|_{\ell_1} \leq b} P_n \ell_{\beta}.$$

Suppose that PY^2 is bounded by a constant and tails of X decay nicely (e.g., $\|X\|_{\infty}$ bounded a.s. or X log concave and isotropic).

Then for increasing d_n and

$$b_n = o \left(\frac{\sqrt{n}}{\log^{3/2} n \log^{3/2}(nd_n)} \right),$$

ℓ_1 -constrained least squares is **persistent**
(i.e., $P \ell_{\hat{\beta}} - P \ell_{\beta^*} \rightarrow 0$).

Application: Persistence

If PY^2 is bounded and tails of X decay nicely, then ℓ_1 -constrained least squares is persistent provided that d_n is increasing and

$$b_n = o \left(\frac{\sqrt{n}}{\log^{3/2} n \log^{3/2}(nd_n)} \right) .$$

Previous Results (Greenshtein and Ritov, 2004):

1. $b_n = \omega(n^{1/2} / \log^{1/2} n)$ implies empirical minimization is not persistent for Gaussian (X, Y) .
2. $b_n = o(n^{1/2} / \log^{1/2} n)$ implies empirical minimization is persistent for Gaussian (X, Y) .
3. $b_n = o(n^{1/4} / \log^{1/4} n)$ implies empirical minimization is persistent under tail conditions on (X, Y) .

Application: Convex Aggregation

Consider $b = 1$, so that the ℓ_1 -ball of radius b is the convex hull of a dictionary of d functions (the components of X).

Tsybakov (2003) showed that, for any aggregation scheme $\hat{\beta}$, the [rate of convex aggregation](#) satisfies

$$P\ell_{\hat{\beta}} - P\ell_{\beta^*} = \Omega \left(\min \left\{ \frac{d}{n}, \frac{d}{n \log d} \right\} \right).$$

For bounded, isotropic distributions, our result implies that this rate can be achieved, up to log factors, by [least squares](#) over the convex hull of the dictionary.

Previous positive results (Tsybakov, 2003; Bunea, Tsybakov and Wegkamp, 2006) involved complicated estimators.

Outline of Talk

1. For ℓ_1 -constrained least squares, bounds on $P\ell_{\hat{\beta}} - P\ell_{\beta^*}$.
 - ▶ **Persistence:**
For what $d_n, b_n \rightarrow \infty$ does $P\ell_{\hat{\beta}} - P\ell_{\beta^*} \rightarrow 0$?
 - ▶ **Convex Aggregation:**
For $b = 1$ (convex combinations of dictionary functions),
what is rate of $P\ell_{\hat{\beta}} - P\ell_{\beta^*}$?
2. For ℓ_1 -regularized least squares, oracle inequalities.
3. Proof ideas.

Proof Ideas: 1. ϵ -equivalence of P and P_n structures

Define

$$G_\lambda = \frac{\lambda}{P(\ell_\beta - \ell_{\beta^*})}(\ell_\beta - \ell_{\beta^*}) : P(\ell_\beta - \ell_{\beta^*}) \geq \lambda \quad .$$

Then:

$\mathbf{E} \sup_{g \in G_\lambda} |P_n g - P g|$ is small

\Rightarrow with high probability, for all β with $P(\ell_\beta - \ell_{\beta^*}) \geq \lambda$,

$$(1 - \epsilon)P(\ell_\beta - \ell_{\beta^*}) \leq P_n(\ell_\beta - \ell_{\beta^*}) \leq (1 + \epsilon)P(\ell_\beta - \ell_{\beta^*})$$

$\Rightarrow P(\ell_{\hat{\beta}} - \ell_{\beta^*}) \leq \lambda$, where $\hat{\beta} = \arg \min_\beta P_n \ell_\beta$.

Proof Ideas: 2. Symmetrization, subgaussian tails

Proof Ideas: 3. Chaining

For a subgaussian process $\{Z_t\}$ indexed by a metric space (T, d) , and for $t_0 \in T$,

$$\mathbf{E} \sup_{t \in T} |Z_t - Z_{t_0}| \leq c \mathcal{D}(T, d) = c \int_0^{\text{diam}(T, d)} \sqrt{\frac{1}{\log N(\epsilon, T, d)}} d\epsilon,$$

where $N(\epsilon, T, d)$ is the ϵ covering number of T .

Proof Ideas: 4. Bounding the Entropy Integral

It suffices to calculate the entropy integral $\mathcal{D}(\sqrt{\lambda}D \cap 2bB_1^d, d)$.
We can approximate this by

$$\mathcal{D}(\sqrt{\lambda}D \cap 2bB_1^d, d) \leq \min \mathcal{D}(\sqrt{\lambda}D, d), \mathcal{D}(2bB_1^d, d) .$$

This leads to:

$$P\ell_{\hat{\beta}} - P\ell_{\beta^*} \leq \frac{c \log^\alpha(nd)}{\delta^2} \min \left(\frac{b^2}{n} + \frac{d}{n}, \frac{b}{\sqrt{n}} \right) \left(1 + \frac{b}{\sqrt{n}} \right) .$$

Proof Ideas: 5. Oracle Inequalities

We get an isomorphic condition on $\{\ell_\beta - \ell_{\beta^*}\}$,

$$\frac{1}{2}P_n(\ell_\beta - \ell_{\beta^*}) - \epsilon_n \leq P(\ell_\beta - \ell_{\beta^*}) \leq 2P_n(\ell_\beta - \ell_{\beta^*}) + \epsilon_n,$$

and this implies that $\hat{\beta} = \arg \min_\beta (P_n \ell_\beta + c\epsilon_n)$ has

$$P\ell_{\hat{\beta}} \leq \inf_\beta P\ell_\beta + c'\epsilon_n .$$

This leads to oracle inequality: For ℓ_1 -regularized least squares,

$$\hat{\beta} = \arg \min_\beta P_n \ell_\beta + \rho_n \|\beta\|_{\ell_1^{d_n}} ,$$

with probability at least $1 - o(1)$,

$$P\ell_{\hat{\beta}} \leq \inf_\beta P\ell_\beta + c\rho_n (1 + \|\beta\|_{\ell_1^{d_n}}) .$$

Outline of Talk

1. For ℓ_1 -constrained least squares,

$$Pl_{\hat{\beta}} - Pl_{\beta^*} \leq \frac{c \log^\alpha(nd)}{\delta^2} \min \left(\frac{b^2}{n} + \frac{d}{n}, \frac{b}{\sqrt{n}} \right) \left(1 + \frac{b}{\sqrt{n}} \right) .$$

- ▶ **Persistence:**

If $b_n = \tilde{o}(\sqrt{n})$, then $Pl_{\hat{\beta}} - Pl_{\beta^*} \rightarrow 0$.

- ▶ **Convex Aggregation:**

Empirical risk minimization gives optimal rate (up to log factors): $\tilde{O} \left(\min(d/n, \sqrt{\log d/n}) \right)$.

2. For ℓ_1 -regularized least squares, oracle inequalities.
3. Proof ideas: subgaussian Rademacher process.